

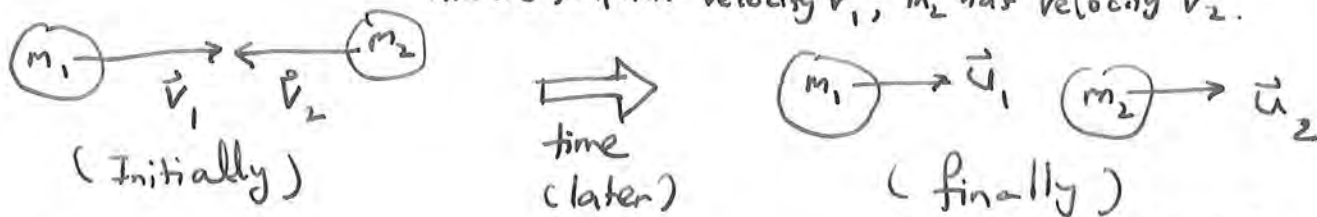
Collisions between objects

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6.1. Conservation of total momentum of a system in collisions

EX1. 2 particles, masses m_1 & m_2 , collide with each other head on (i.e. they're directly heading towards each other.)
Initially, m_1 has velocity \vec{v}_1 , m_2 has velocity \vec{v}_2 .



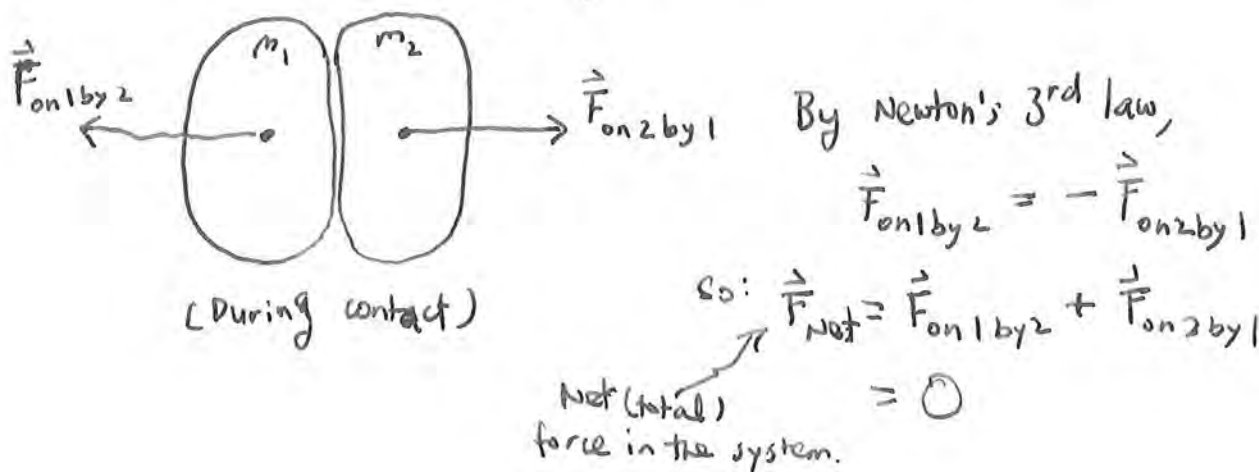
After they collide, m_1 moves with velocity \vec{u}_1 and m_2 moves with velocity \vec{u}_2 . [after collision]

Qu: Determine \vec{u}_2 in terms of all other parameters of the system (i.e. $m_1, m_2, \vec{v}_1, \vec{v}_2, \vec{u}_1$).

Sol'n: Initially, (before collision); $\vec{P}_{\text{total before}} = m_1 \vec{v}_1 + m_2 \vec{v}_2$
total momentum of system before collision.

After the collision, the system's total momentum is: $\vec{P}_{\text{total after}} = m_1 \vec{u}_1 + m_2 \vec{u}_2$.

Now, the key is to analyze what force each particle feels during the collision. (i.e. while the particles are in contact with each other during the brief amount of time.)



And note that $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{net}$; where \vec{P}_{total} = total momentum of system at any time

• During the collision, we have $\frac{d\vec{P}_{total}}{dt} = 0$.

• So, this means that $\vec{P}_{total}^{before} = \vec{P}_{total}^{after}$ [i.e. The total momentum ~~has~~ never changes during the collision.]

Thus: $m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{u}_1 + m_2 \vec{u}_2$

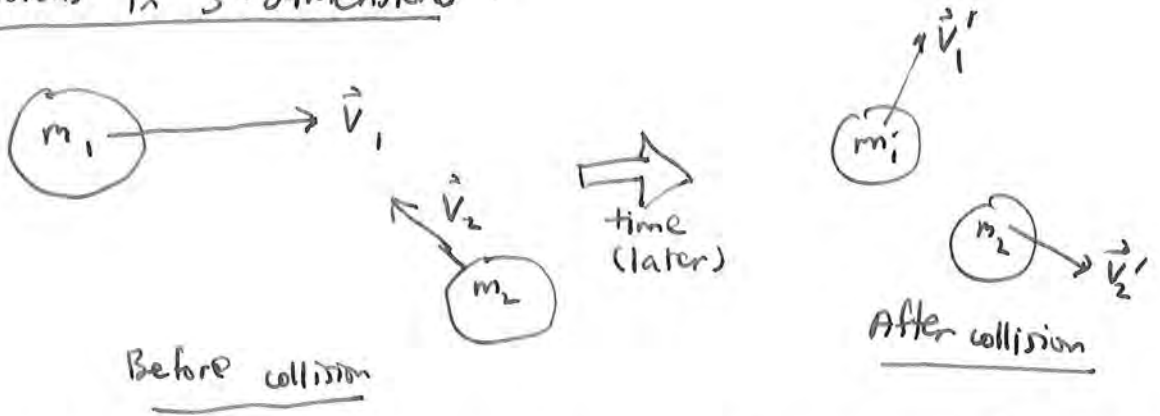
$\Rightarrow \frac{m_1 (\vec{v}_1 - \vec{u}_1) + m_2 \vec{v}_2}{m_2} = \vec{u}_2$

← velocity of m_2 after the collision, expressed in terms of all other variables.

In the previous example, we constrained ourselves to 1-dimension (head-on collisions). But the same reasoning should also work in 3-dimensional motion (i.e. Not just head-on collisions.)

Let's repeat the reasoning:

Ex 2 Collisions in 3-dimensions:

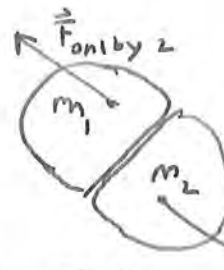


• Before collision: $\vec{P}_{total} = m_1 \vec{v}_1 + m_2 \vec{v}_2$
total momentum of system

• After collision: $\vec{P}'_{total} = m_1 \vec{v}'_1 + m_2 \vec{v}'_2$

Note that: 1.) Before collision: $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{total} = 0$ ← No net force in system because no force on either particle
2.) After collision: $\frac{d\vec{P}'_{total}}{dt} = \vec{F}'_{total} = 0$ ←

And 3.) During the collision:



$$\vec{F}_{1on2} = -\vec{F}_{2on1}$$

↳ Newton's 3rd law.

$$\text{So } \frac{d\vec{P}_{\text{total (during collision)}}}{dt} = \vec{F}_{\text{net (during collision)}} = 0$$

(because $\vec{F}_{\text{net (during collision)}} = \vec{F}_{1on2} + \vec{F}_{2on1} = 0$)

So putting 1.), 2.) & 3.) together, we see that at no point during the whole process (before, during, and after collision), is there ever any net (total) force in the system.

∴ Total momentum remains constant during the whole process.

We say that "the total momentum is conserved during collision"

$$\text{Thur: } m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}'_1 + m_2\vec{v}'_2$$

- EX3: In the previous example, (a) What is the velocity of the center of mass (CM) before the collision?
 (b) After the collision?
 (c) How does the position of CM change over time? (see above)

Sol'n: (a) Before the collision: $\vec{V}_{\text{cm}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

(b) After the collision: $\vec{V}'_{\text{cm}} = \frac{m_1\vec{v}'_1 + m_2\vec{v}'_2}{m_1 + m_2}$

← CM velocity before and after collision.

⚡ Important: Note that since $\frac{d\vec{P}_{\text{total}}}{dt} = 0$ at all times, we have

$$m_{\text{total}}\vec{V}_{\text{cm}} = m_{\text{total}}\vec{V}'_{\text{cm}}$$

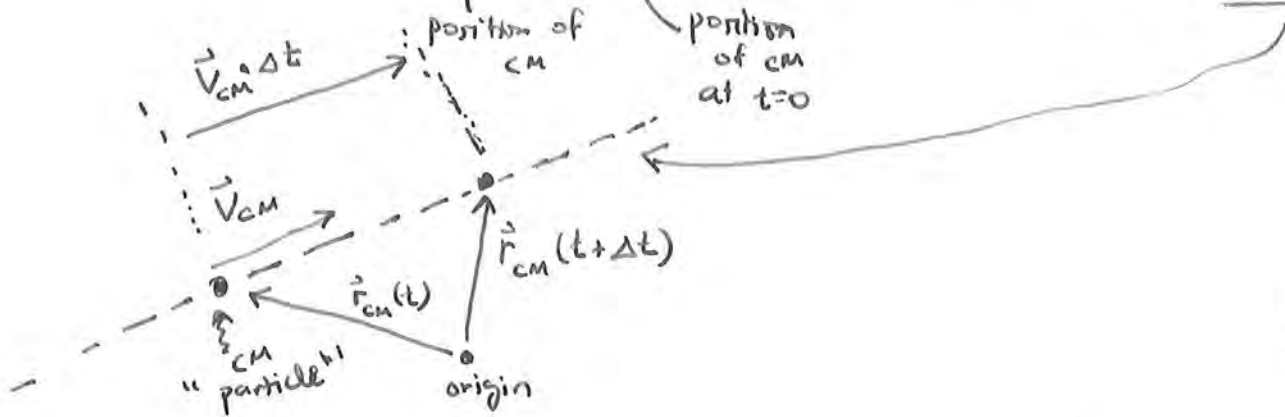
$$\Rightarrow \boxed{\vec{V}_{\text{cm}} = \vec{V}'_{\text{cm}}}$$

(due to conservation of total momentum)

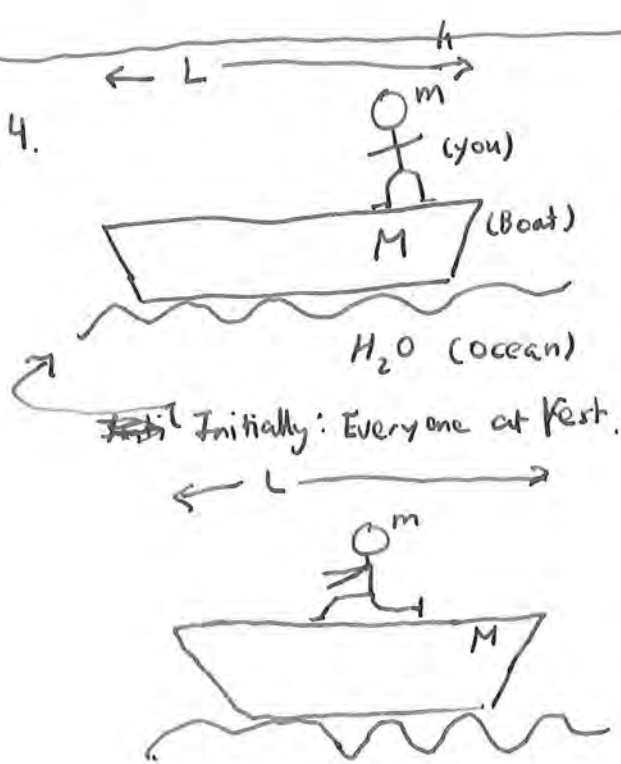
(c) How does the CM's position change over time? (pg 6-4)

Sol'n: from (b) & (a) [due to the conservation of the total momentum], we see that the CM (point-particle that represents the whole system) moves at all times with constant velocity $\vec{V}_{cm} (= \vec{V}'_{cm})$.

so: $\vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{V}_{cm} \cdot t$ (Constant velocity motion in a straight line)



Ex 4.



Initially, you and the boat are at rest on a calm ocean water.

Assume there's no friction between the ocean and boat, and between you and the boat. You stand at the end of the boat at $t=0$. Boat has length L .

You start to walk to the ~~left~~ ^{left end} of the boat. Sometimes you walk at a fast speed, sometimes you walk at a slower speed. When you reach the left end of the boat, you stop.

Questions

- (A) What's happening to the boat as you're walking?
- (B) What happens to the boat when you stop moving after reaching the left end of the boat?
- (C) Does the boat move at all? If so, by how much and in which direction?

sol'n (A) It might seem like this is a challenging question because the person is moving with a speed that's varying over time (sometimes walking fast, sometimes slow.) But this question becomes easy if you just think about the main principles that are underlying the phenomenon here.

The main principles here are: 1) Internal forces add up to zero (i.e. they cancel each other)
 ↑ covered in the previous lecture.
 and 2) $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{net}$. (And $\vec{F}_{net} = \vec{F}_{external} + \vec{F}_{internal}$)
 ↑ total external force ↑ total internal force
 Covered in previous lecture. || 0

Note that $\vec{F}_{net} = 0$ here because there are no external forces in the system

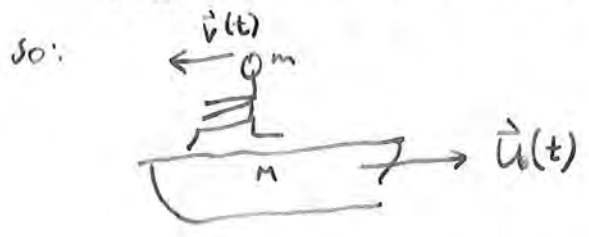


System = you + boat.
 Here, internal forces are: $\vec{F}_{on\ boat\ by\ you}$ and $\vec{F}_{on\ you\ by\ boat}$.
 By Newton's 3rd law: $\vec{F}_{on\ you\ by\ boat} = -\vec{F}_{on\ boat\ by\ you}$.

So; $\frac{d\vec{P}_{total}}{dt} = 0$ at all times.

(Here, $\vec{P}_{total} = \vec{P}_{you} + \vec{P}_{boat}$)

And since $\vec{P}_{initial} = 0$ (No one is moving), we must have $\vec{P}_{total} = 0$ at all times



$$m\vec{v}(t) = -M\vec{u}(t)$$

where $\vec{v}(t)$ = velocity of person at time t
 $\vec{u}(t)$ = Boat's velocity at time t

Note that boat must move to right to cancel ~~you~~ out ~~you~~ the person's momentum.

(B) When you stop walking, we have $\vec{v}(T) = 0$
at $t=T$

(pg 6-6)

$$m\vec{v}(T) = 0 \\ = -M\vec{u}(T) \Rightarrow \vec{u}(T) = 0 \Rightarrow \text{Boat also stops moving at the same time as you.}$$

(c) The boat has moved to the right by your walking to the left. When you stop, the boat stops too. Where ~~is~~ is it when it stops?
~~Boat~~, ~~Person~~

The concept that connects the positions of individual objects in a system is the concept of center of mass.

Recall from the last lecture that the position of center of mass is the "weighted average" of the positions of the individual components of the system. (Here, they are the person & the boat.)

Namely, $X_{cm} = \frac{mX_{person} + MX_{boat}}{m+M}$
↑
position of cm

At $t=0$ (initial time), no one is moving. So $\vec{p}_{total}(t=0) = 0$

Here, we're using the fact that

the ~~cm~~ momentum of cm $= \vec{p}_{total}$

$$\Rightarrow (m+M) \vec{v}_{cm}(t=0) = 0$$

(see ex 11 on pg 5-30)

so $\vec{v}_{cm}(t=0) = 0$

and since $\frac{d\vec{p}_{total}}{dt} = 0$ at all times, $\vec{v}_{cm}(t) = 0$ at all times (see (B).)

Thus, $X_{cm}(t) = X_{cm}(t=0)$

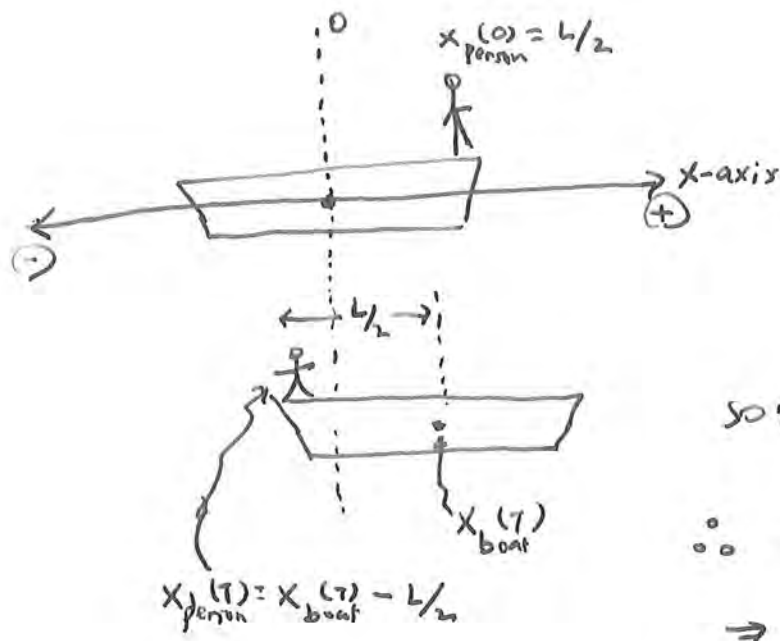
↑
Never moves even though the boat & the person move.

so: $X_{cm}(t) = 0 = \frac{mX_{person}(t) + MX_{boat}(t)}{m+M}$

$$\Rightarrow -mX_{person}(t) = MX_{boat}(t)$$

At $t=0$, ~~person~~ ~~boat~~

→
over



$$\text{so: } x_{\text{person}}(T) = x_{\text{boat}}(T) - L/2$$

$$\therefore -m x_{\text{person}}(T) = M x_{\text{boat}}(T)$$

$$\Rightarrow -m [x_{\text{boat}}(T) - L/2] = M x_{\text{boat}}(T)$$

$$\Rightarrow \boxed{x_{\text{boat}}(T) = \frac{mL/2}{m+M}}$$

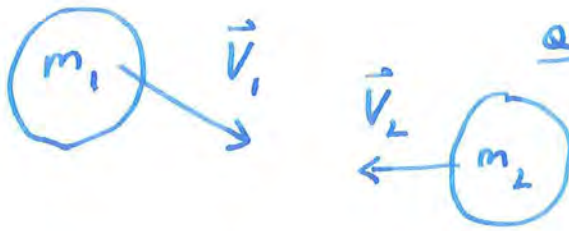


6.2. Working in the center of mass frame

(pg 6-8)

- Center of mass frame: As an observer, you're moving with the same velocity as the center of mass of the system. Analyzing collisions while you're in the CM frame makes calculations easier.

Ex 5



Qn: How does this collision look like if you're an observer in the CM reference frame?

Sol'n:
$$\vec{V}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \leftarrow \text{velocity of CM.}$$

Now, suppose you're on a car that moves ~~with~~ ^{at} \vec{V}_{cm} . What do you see?

Relative to you, the 1st particle with mass m_1 moves ~~with~~ ^{at} velocity: $\vec{v}_1 - \vec{V}_{cm}$ (call this \vec{u}_1)

Relative to you, the 2nd particle with mass m_2 moves at velocity: $\vec{v}_2 - \vec{V}_{cm}$ (call this \vec{u}_2)

so, ~~the~~ before the collision, you'll see:



$$\vec{P}_{total} \text{ before} = m_1 \vec{u}_1 + m_2 \vec{u}_2$$

~~XXXXXX~~

Now note that:

(pg 6-9)

$$\vec{p}_{\text{total before}} = m_1 \vec{u}_1 + m_2 \vec{u}_2$$

$$= m_1 (\vec{v}_1 - \vec{v}_{\text{cm}}) + m_2 (\vec{v}_2 - \vec{v}_{\text{cm}})$$

$$= m_1 \vec{v}_1 + m_2 \vec{v}_2 - m_1 \vec{v}_{\text{cm}} - m_2 \vec{v}_{\text{cm}}$$

$$= m_1 \vec{v}_1 + m_2 \vec{v}_2 - (m_1 + m_2) \vec{v}_{\text{cm}}$$

$$= m_1 \vec{v}_1 + m_2 \vec{v}_2 - \cancel{(m_1 + m_2)} \left[\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{\cancel{m_1 + m_2}} \right]$$

$$= 0$$

⇒ As an observer in the CM reference frame, you see that the total momentum of the system before collision is Zero.

We could also get the same result by noting that:

$$\vec{v}_{\text{cm}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\Rightarrow 0 = \vec{v}_{\text{cm}} - \vec{v}_{\text{cm}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} - \vec{v}_{\text{cm}}$$

$$\Rightarrow 0 = m_1 (\underbrace{\vec{v}_1 - \vec{v}_{\text{cm}}}_{\vec{u}_1}) + m_2 (\underbrace{\vec{v}_2 - \vec{v}_{\text{cm}}}_{\vec{u}_2})$$

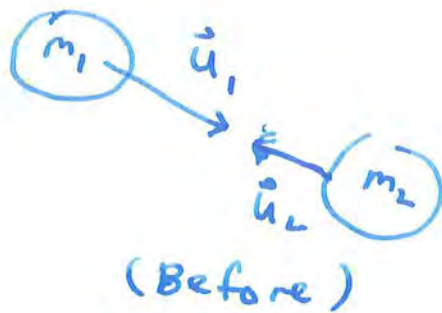
$$\Rightarrow 0 = m_1 \vec{u}_1 + m_2 \vec{u}_2 \\ = \vec{p}_{\text{total initial}}$$

$$\text{so } \boxed{-m_1 \vec{u}_1 = m_2 \vec{u}_2}$$

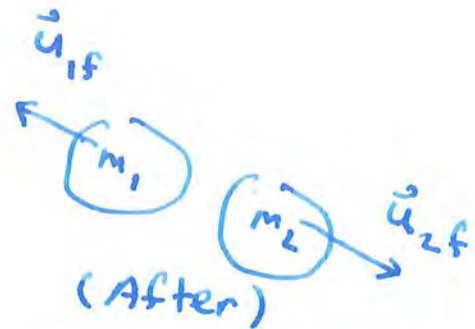
To you (an observer in the CM frame), the collision still preserves the system's total momentum.

Thus: $\vec{P}_{\text{final total}} = 0$ ← system's total momentum as seen by an observer in the CM ref. frame.

So:



$$-m_1 \vec{u}_1 = m_2 \vec{u}_2$$



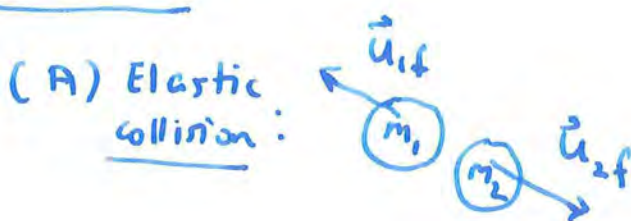
$$-m_1 \vec{u}_{1f} = m_2 \vec{u}_{2f}$$

These Equations are simpler if you work in the CM frame (i.e. there are less terms in the equations.)

There are several possible collisions:



After collision:



$$\left. \begin{aligned} \vec{u}_{1f} &= -\vec{u}_1 \\ \vec{u}_{2f} &= -\vec{u}_2 \end{aligned} \right\} \Rightarrow \text{Total kinetic energy unchanged after collision}$$



$$\left. \begin{aligned} \vec{u}_{1f} &= 0 \\ \vec{u}_{2f} &= 0 \end{aligned} \right\} \Rightarrow \text{so } \vec{P}_{\text{final total}} = 0$$

And total KE after collision = 0 in the CM frame.