MITES 2008 : Physics III - Oscillations and Waves :: Final Examination.

Massachusetts Institute of Technology
Instructor: Hyun Youk
Recitation Instructor: Louis Fouche
(Wednesday, July 30, 2008 : 9:00 - 11:00 AM.)

You have **two hours** to complete all the questions and problems in this examination. This examination consists of **two sections**:

Section 1: Short Questions [Total: 50 points]: For all these questions, you only need to supply either a few sentences, perhaps one or two equations, or a simple diagram with one or two sentences are sufficient to obtain a full credit.

Section 2: Problems [Total: 80 points]: Involves calculations and tests your skill in using mathematics to model physical systems as well as your physical intuition.

This exam is thus worth **130 points**. But I will grade your exam out of **110 points**. This means that your goal is to get either 110 points or over on this exam by solving as many short questions and problems as you can.

Since this is a timed exam, your solutions do not have to be as "organized" as your problem set solutions. You do not have to show any more work than what is necessary for you to get your answers. But be warned: if you're unsure about your solution and you want some partial credit, you do need to have written down some work.

Unless the question asks you to "derive" a solution, you can just state what the solution is (eg. "Write the solution to the EOM" means you can just state the solution without derivation).

The following formulas may be useful:

$$exp(i\alpha t) = cos(\alpha t) + isin(\alpha t)$$

$$\text{Real part of} \quad Aexp(i\omega t) + Bexp(-i\omega t) \quad \text{is} \quad Ccos(\omega t - \phi)$$

$$Ccos(\omega t - \phi) = Ccos(\omega t)cos(\phi) + Csin(\omega t)sin(\phi)$$

$$\text{Quadratic equation:} \quad ax^2 + bx + c = 0 \quad \Rightarrow \quad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$cos^2(\theta) + sin^2(\theta) = 1$$

$$sin(-\theta) = -sin(\theta) \qquad cos(-\theta) = cos(\theta)$$

$$\text{Integral for Fourier series:} \quad \frac{2}{L}\int_0^L sin\Big(\frac{n\pi x}{L}\Big)sin\Big(\frac{m\pi x}{L}\Big)dx = 0 \qquad \text{(if n} \neq \text{m)}$$

$$= 1 \qquad \text{(if n = m)}$$

$$\text{Complex representation of plane wave moving to the right:} \quad Ae^{i(kx-\omega t)}$$

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I. SHORT QUESTIONS [TOTAL: 50 PTS]

Q1. [3 points] : What is a simple harmonic oscillator? (Hint: A simple harmonic oscillator does *not* need to be a block attached to a spring). Your answer should be short and precise. Write down an equation if you can.

Q2. [3 points] : What is a damped simple harmonic oscillator? (Again, as with Q1, your answer to this should be short and precise. And again, there doesn't have to be a spring). Furthermore, what physical parameters determine the three different regimes of damping?

Q3. [2 points] : In an underdamped simple harmonic oscillator, does the oscillator's angular frequency decrease or increase over time? What about its amplitude?

Q4. [5 points] : What happens when an undamped (i.e. frictionless) simple harmonic oscillator (SHO) is subjected to a sinusoidally oscillating external force (with angular frequency ω) which is much larger than the natural angular frequency ω_0 of the SHO (i.e. $\omega \gg \omega_0$)? It's sufficient to give a rough sketch of graph showing the oscillator's position $x(t)$ as a function of time t. Justify your sketch using physical intuition. (Hint: You shouldn't have to do any calculation to get this plot, physical intuition is sufficient.)

Q5. [5 points] : Same question as Q4, but this time, sketch roughly the oscillator's position $x(t)$ as a function of time t when $\omega \ll \omega_0$. Justify your sketch using physical intuition.

Q6. [2 points] : Given a coupled oscillator made up of N particles, how many equations of motion describe this system in its entirety? How many free parameters exist in your most general solution? (Here, "general solution" refers to a solution describing any arbitrary motion of these N particles).

Q7. [6 points] : As in Q5, consider a coupled oscillator containing N particles. If $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ describe the displacement of each of the N particles from equilibrium, then how do we go about finding the normal modes of the system (i.e. what function shall we use as our guess for $x_j(t)$, or equivalently its complex-number equivalent $z_j(t)$?). How many normal modes are there for N particles? The general solution is just a sum of all the normal modes. Give a brief explanation of the principle that allows you to add up these solutions to get the general solution describing the motion of the N particles.

Q8. [5 points] : Describe how you would go about deriving the wave equation from a system of coupled oscillators involving N particles. What approximations do you need to use? (Be sure to mention what a continuum limit is).

Q9. [3 points] : In class, we showed that *any* one variable function $f(z)$ can be turned into a solution to the one-dimensional wave equation as long as we substitute in $z = x \pm vt$. That is, $y(x,t) = f(x \pm vt)$ satisfies the wave equation. If this is the case, then how come $y(x,t) = Aexp(k(x - vt))$ is *not* a solution to the wave equation when describing a string with its ends fixed at two walls that are separated by length L?

Q10. [4 points] : Is $\mathrm{Aexp}(\mathrm{kx}^2)\mathrm{exp}(\mathrm{k}(\mathrm{vt})^2)$ a solution to the wave equation without boundary conditions (i.e. no walls)? Here, A, k, and v are real number constants. If it's a solution, why? If not, why isn't it a solution? Is it describing a periodic wave? If so, what is its wavelength?

Q11. [4 points] : Show, using a simple physical argument, that electromagnetic wave (or some signal) must travel from a moving charged point particle.

Q12. [3 points] : Why can't an electromagnetic wave penetrate a good metal conductor?

Q13. [5 points] : Consider a simple harmonic oscillator composed of a block of mass $1kg$ attached to a spring. It's equilibrium position is at $x = 0$ and it's set to oscillate about this equilibrium position. If we measure the instantaneous speed of the block, as it passes by $x = 0$ to be 2 m/s, what is the total energy of the system? If the SHO's amplitude is 1 m, what is the spring constant k? Be sure to include units in your answer.

II. PROBLEMS (TOTAL: 80 PTS)

Problem 1. Simple Harmonic Oscillator. {Total = 30 points}

Consider a block of mass m attached to two Hookian springs with stiffness k and $3k$ respectively on an inclined plane as shown in Figure 1. One spring is fastened to a wall at the top of the ramp while the other spring is fastened to a wall at the bottom of the ramp. The angle of inclination is θ. The *displacement* of the block measured parallel to the incline and relative to the block's equilibrium position is s. $s = 0$ denotes the equilibrium position of the block. Acceleration due to gravity is a constant g. Assume that the incline is *frictionless*.

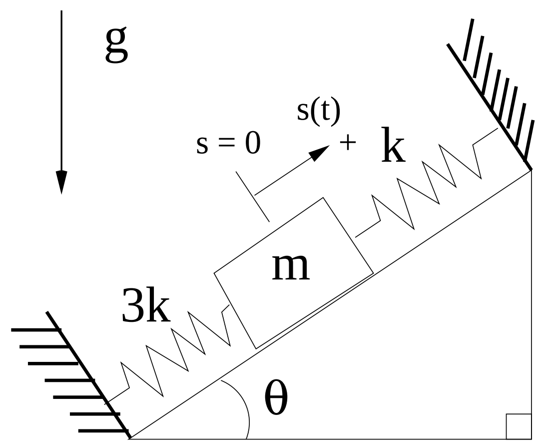

FIG. 1: A block attached to two springs on a *frictionless* inclined plane. The two springs have stiffness k and $3k$ respectively. One spring is fastened to a wall at the top of the ramp while the other spring is fastened to a wall at the bottom of the ramp. The ramp is inclined with an angle θ. The block has mass m and the acceleration due to gravity is a constant g.

(a.) [5 points] : Derive the equation of motion describing the block, thus showing that the block acts as a simple harmonic oscillator. What is the natural angular frequency ω_0 of the block? You should find that the equation of motion is

$$\frac{d^2 s}{dt^2} + \omega_0^2 s = 0 \tag{1}$$

(b.) [2 points] : Write down the general solution to the equation of motion you derived in (a.). Define what the physical meaning of all your free parameters in your solution are. (When you're told to "write down", it means state, not derive. Of course, you can derive the general solution by the "guess and check" method in this case.)

(c.) [3 points] : Write down the total energy of the system above. The system consists of the block and the springs so the total energy should involve all three objects.

(d.) [4 points]: From the total energy, derive the equation of motion. You should get the same equation as the one obtained in (b.). Be sure to justify each step of your derivation.

(e.) [4 points]: Suppose we roughen the surface of the incline so that there's now a damping force on the block. The damping force acts parallel to the inclined plane, with force F = -b$\dot{s}$ (b is the *damping constant*). Show that the equation of motion is now

$$\frac{d^2s}{dt^2} + 2\gamma\frac{ds}{dt} + \omega_0^2 s = 0 \tag{2}$$

where γ is a constant you need to determine in terms of the parameters given in this problem.

(f.) [5 points]: Solve the complex number version of the damped simple harmonic motion equation you derived in (e.). Write down the *general* solution in its complex number form. Looking at your solution, state the condition under which the moving block comes to rest the fastest, without oscillating.

(g.) [3 points]: If the block is undergoing underdamped oscillation, what is the maximal heat energy that can be delivered to the ramp after time $\Delta t = \frac{2}{\gamma}$? Assume that the block starts out with an amplitude of A at $t = 0$. Recall that the amplitude decays over time in an underdamped oscillation.

(h.) [4 points]: Suppose now the ramp's angle of inclination θ is a function of time ($\theta = \theta(t)$) and that the ramp is frictionless again. In particular, $\theta(t) = \theta_0 cos(\omega t)$ where θ_0 is a very small constant ($|\theta_0| \ll 1$)? What would happen to the motion of the block? (Hint: It may help you to remember that $sin(\phi) \approx \phi$ when $|\phi| \ll 1$). In addition,
If $\dfrac{2\pi}{\omega} = 2$ years and $\dfrac{2\pi}{\omega_0} = 1$ second, approximate the period of oscillation of the block.

Problem 2. Freely propagating wave. {Total = 20 points}

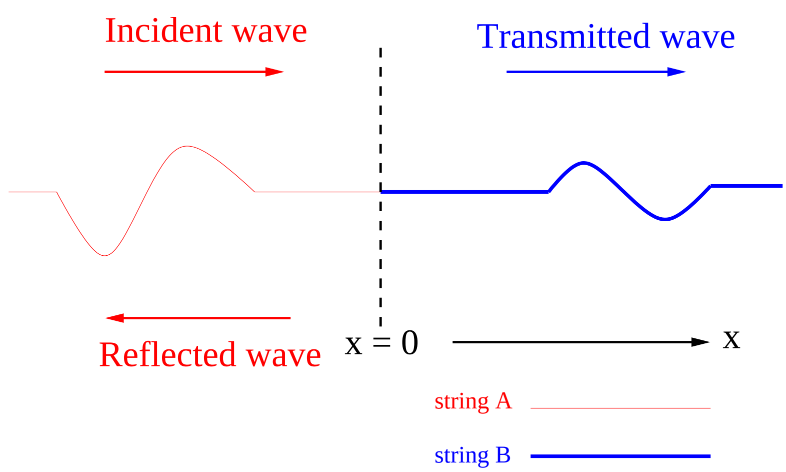

FIG. 2: . **This figure is pertinent only to problem 2 (c) through (g)**. Incident and reflected waves travel in string A, while transmitted wave travels in string B. String A and B have mass density ρ_1 and ρ_2 respectively. Both are under constant tension T. In string A, wave travels at speed c_1 while in string B, wave travels at speed c_2.

(a.) [5 points]: Derive the most general solution to a one-dimensional wave equation. To remind you, the wave equation in one dimension is

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}, \tag{3}$$

where v is the wave speed. *(Hint: Think of a pulse moving to the right, and a pulse moving to the left with speed v).* Your answer should describe what *form* of $f(x,t)$ is a solution rather than specifying the exact expression of $f(x,t)$. Namely, you should derive that the most general solution is a superposition of a left moving and a right moving pulse, both with speed v.

(b.) [3 points]: Write down the complex-number version of plane wave, moving to the right with speed $\frac{v}{2}$. Your answer should be in terms of wave number $k = \frac{2\pi}{\lambda}$, wave speed $\frac{v}{2}$, and complex number amplitude $\tilde{A}$. What is the corresponding wave equation that this complex plane wave is a solution to?

(c.) [3 points]: Next, write down the complex number representations of incident, reflected, and transmitted waves in the case of two strings (string A and string B) joined to each other at a junction (located at $x = 0$). The setup is shown in Figure 2.

(d.) [3 points]: Write down the expression for the (complex number representation) of the *resultant wave* you'd see with your eyes in string A, and in string B. Let $y_1(x, t)$ be the profile of the string A and $y_2(x, t)$ be the profile of string B.

(e.) [3 points]: Write down the two boundary conditions at the junction $x = 0$. For each boundary condition (represented by an equation) you write down, be sure to state one sentence describing what the equation *physically* represents. Applying these boundary conditions to $y_1(x, t)$ and $y_2(x, t)$, what are the two equations that describe the relationships among the three complex number amplitudes?

(f.) [3 points]: Based on physical intuition, what would happen to the transmitted wave when string B has mass density much larger than the mass density of string A? Based on this answer and the relationships among the three complex amplitudes you found in (f.), what is the relationship between the amplitudes of incident and reflected waves?

Problem 3. Fourier series. {Total = 30 points}

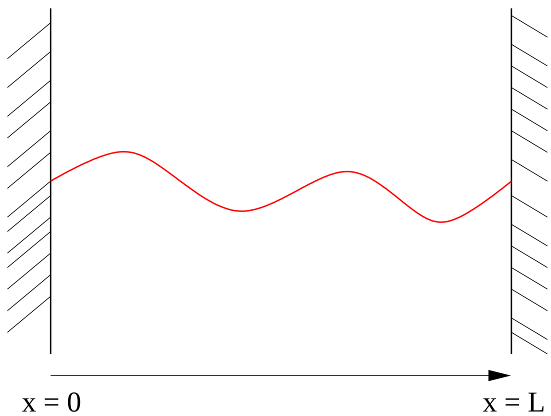

FIG. 3: String between two fixed walls.

(a.) [10 points] : By solving the wave equation obeyed by a string with its ends fixed at two walls separated by length L, show that the normal modes of the string are standing waves. For each normal mode describing a standing wave, determine what the wave number k, wave length λ, and the normal mode angular frequency ω are. Be sure to specify what the boundary conditions are when you're solving the wave equation. Namely, show that the normal modes are

$$y_n(x,t) = \{A_n cos\Big(\frac{n\pi vt}{L}\Big) + B_n sin\Big(\frac{n\pi vt}{L}\Big)\} sin\Big(\frac{n\pi x}{L}\Big), \tag{4}$$

where n can be any positive integer.

(b.) [4 points] : What is the most general shape of string $y(x, t)$ that can be supported between the two walls? Express your answer in terms of the normal modes you found in (a.).

(c.) [10 points] : Suppose the string initially (at $t = 0$) has the shape described by $g(x)$:

$$g(x) = \frac{L}{4} sin\Big(\frac{8\pi x}{2L}\Big) - \frac{L}{3} sin\Big(\frac{18\pi x}{L}\Big) \qquad (\ 0 < x < L \ ) \tag{5}$$

and that initially (t =0), the string has the velocity profile given by $h(x)$:

$$h(x) = \frac{v_0}{5} sin\Big(\frac{16\pi x}{L}\Big) + \frac{v_0}{2} sin\Big(\frac{2\pi x}{L}\Big) \qquad (\ 0 < x < L\ ) \tag{6}$$

What is the shape of string $y(x,t)$ for subsequent time $(t > 0)$? (*Hint*: If you're finding yourself solving complicated integrals, think again. While you can get the answer by doing the right integral, you can also get the answer much more easily by looking at $g(x)$ and $h(x)$ very carefully and thinking about the property of the integral $\frac{2}{L}\int_0^L sin(\frac{n\pi x}{L}) sin(\frac{m\pi x}{L}) dx$.) No need for a heavy evaluation of any integrals but you do need to justify your answer.

(d.) [6 points] : Suppose now that $g(x)$ and $h(x)$ describe the initial conditions $(t = 0)$ but for a string whose ends are fixed between two walls that are distance $\frac{L}{2}$ apart. That is, the initial shape of string is now

$$g(x) = \frac{L}{4} sin\Big(\frac{8\pi x}{2L}\Big) - \frac{L}{3} sin\Big(\frac{18\pi x}{L}\Big) \qquad (\ 0 < x < \frac{L}{2}\ ) \tag{7}$$

and the initial velocity profile of the string is

$$h(x) = \frac{v_0}{5} sin\Big(\frac{16\pi x}{L}\Big) + \frac{v_0}{2} sin\Big(\frac{2\pi x}{L}\Big) \qquad (\ 0 < x < \frac{L}{2}\ ) \tag{8}$$

What is the shape of string $y(x,t)$ for subsequent time $(t > 0)$? Again, no need for a heavy evaluation of any integrals. But you do need to justify your answer.