MITES08 : Physics III : Final Exam Solutions 7/30/08

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(pg 1)

A : Short Questions Section

Q1.) SHO is any system whose eq'n of motion (EOM) is $\ddot{x} + \omega_0^2 x = 0$ where $x(t)$ is the position of particle and ω_0 = angular frequency of oscillation

Q.2.) Damped SHO is a system whose EOM is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0;$$

where $x(t)$ = position of particle, 2γ measures "strength" of damping, and ω_0 is angular frequency the system would oscillate with in absence of damping.

(SHO + damping gives damped SHO.)

Q3) In under-damped SHO,

- Amplitude decreases over time

But

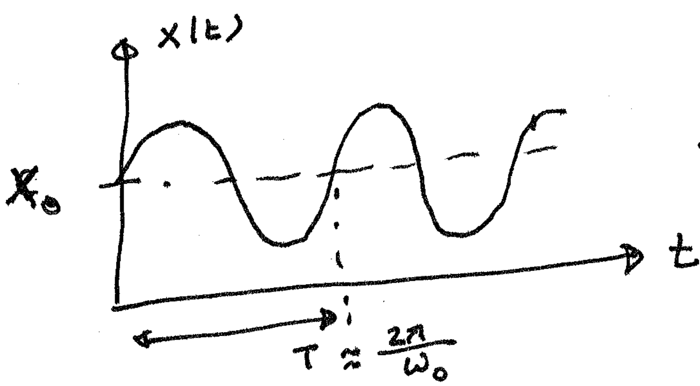
- Angular frequency remains constant over time.

Q4) SHO subjected to $F(t) = C_0\cos(\omega t) + C_0$

external force

$\omega \gg \omega_0$

Natural angular frequency.



← Fast F(t) oscillation Averages to a constant

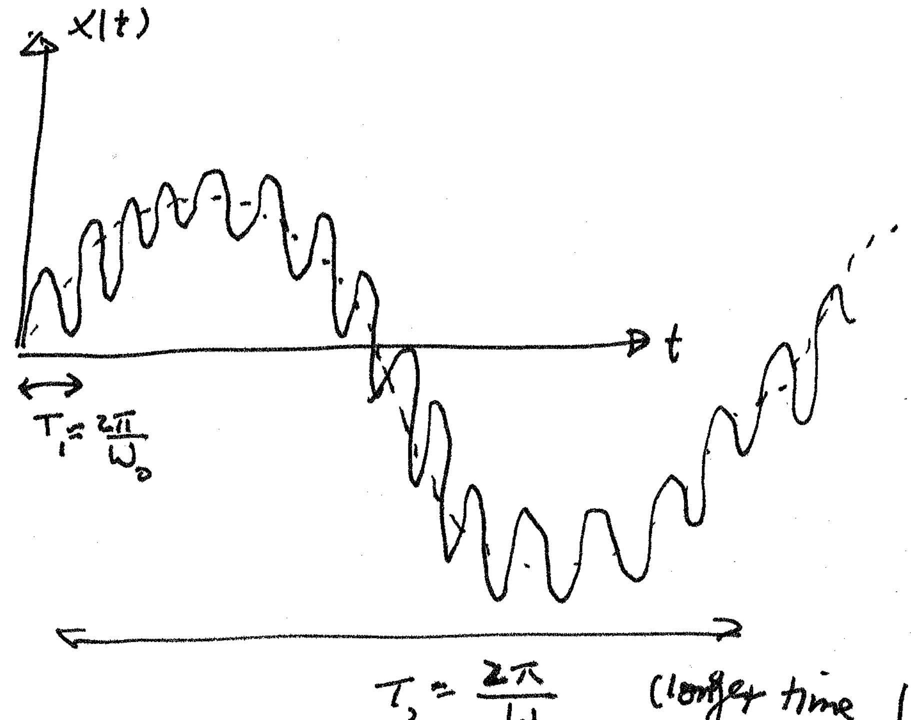
⇒ Like a constant force acting on a particle

⇒ Doesn't affect frequency of oscillation

still ω_0.

(Q5) $F(t) = C_0 + C_1 \cos(\omega t)$ $\omega \ll \omega_0$

Slowly varying sinusoidal force.



- On a short time scale, looks like ~ constant force.
  Only on a longer time scale, see the effect of $F(t)$ oscillating with angular frequency ω.

(Q6.) System of N eq'ns of motion describe coupled oscillators of N particles.

- There are $2 \times N$ free parameters in the general sol'n.

Q7.) To find the normal modes:

Guess: $\begin{pmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_N(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} e^{i\omega t}$

$\omega \equiv$ Normal mode angular frequency.

Column vector of N constants.

Then plugging this into the system of N EOMs, we can find ω.

- For N particles $\Rightarrow$ N normal modes.
- General sol'n $= \sum_{n=1}^{N} (\text{Normal mode})_n$

Linearity of this system of EOMs allows us to do this

Linearity means that if $y_1(t)$ and $y_2(t)$ are each individually solution to the ~~system~~ EOM, then so is $y_1(t) + y_2(t)$.

Q8.) · To derive wave eq'n from a coupled oscillator of N particles (e.g. slinky of N particles),

· Write down EOM for each particle. (from Newton's 2nd law).
(e.g. Apply Newton's 2nd law to j-th particle)

· Then take the continuum limit to turn the EOMs to a single wave eq'n.

- Continuum limit: $N \to \infty$. (Take N large) while keeping the total length L of the system fixed and finite.

$\Rightarrow$
$$a = \frac{L}{N} \underset{(N\to\infty)}{\Rightarrow} a \to \text{infinitesimal} \Rightarrow \underline{a \equiv \delta x}$$

distance between 2 adjacent particles in equilibrium in the slinky.

Then the finite difference eq'n of the form $\frac{y_j(t) - y_{j-1}(t)}{a}$

becomes $\frac{\partial y(x_j, t)}{\partial x}$.

~~Sig~~ Also, we can then drop the subscript "j"

and write $\underset{\text{discrete variable}}{x_j} \longrightarrow \underset{\text{continuous variable}}{x}$ because $x_{j+1} - x_j = a = \delta x$.

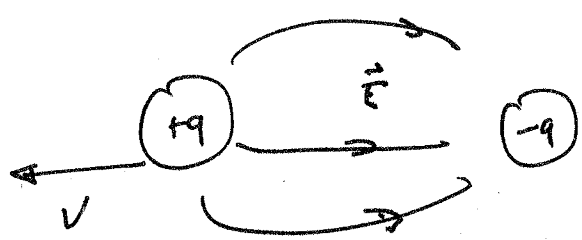
Q9.) $y(x,t) = A\exp(k(x-vt))$ is not soln for a string between 2 walls. because it doesn't satisfy the Boundary conditions $y(x=0,t)=0$
and $y(x=L,t)=0$.

Q10) No, not a sol'n to wave eq'n. even w/o boundary conditions. because you cannot get $Ae^{kx^2}e^{k(vt)^2}$ just by using the substitution $z = x \pm vt$

i.e. Not of the form $y(x,t) = f(x \pm vt)$.

- It's not describing a periodic wave thus.

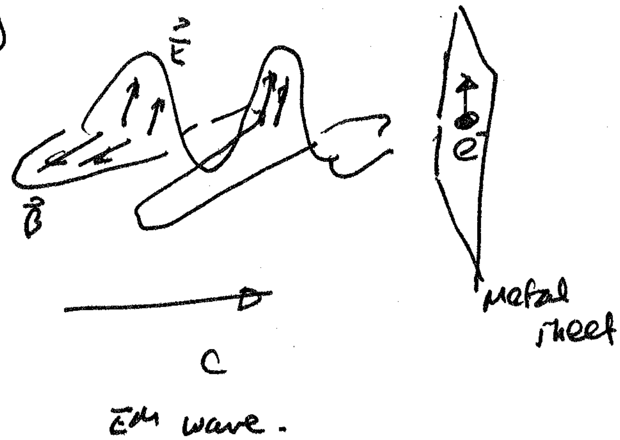
Q11)



Move this charge. Then this charge must somehow "feel" displacement of "+q". Takes finite time to travel (Not instantaneously) from +q to -q charge.

$\Rightarrow$ EM wave generated.

Q12.)



$\vec{E}$ & $\vec{B}$ components of EM wave does work on free electrons in the metal.

$\Rightarrow$ Energy is transferred to e^-.

$\Rightarrow$ Wave dies away.

Q13)

V_{max} occurs at $x=0$. (spring ~~unstretched~~) in equilibrium) configuration when $x=0$.

$V_{max} = 2\,m/s$.

$$E_{tot} = KE_{max} = \frac{mV_{max}^2}{2} = \frac{(1\,kg)\,(2\,m/s)^2}{2} = \boxed{2\,J.}$$

$$E_{tot} = \frac{KA^2}{2} \Rightarrow 4J = KA^2$$

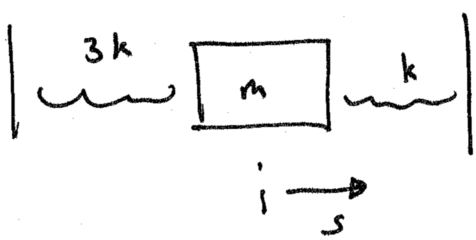
$$\Rightarrow K = \frac{4}{1}\,N/m \Rightarrow \boxed{K = 4\,N/m}$$

II. Problems

P1

1(a) Ignore gravity since $s(t) \equiv$ displacement from equilibrium

(i.e. $s=0$ equilibrium).



$$m\ddot{s} = -ks - 3ks$$

$$\Rightarrow \ddot{s} + \left(\frac{4k}{m}\right)s = 0$$

$$\Rightarrow \boxed{\ddot{s} + \omega_0^2 s = 0} \qquad \omega_0 = \sqrt{\frac{4k}{m}}$$

Natural angular frequency

1(b) General soln : $\boxed{s(t) = S_0 \cos(\omega_0 t - \phi)}$

2 Free parameters : $S_0 \equiv$ Amplitude

$\phi \equiv$ phase shift.

1(c)

$$E_{tot} = \frac{m\dot{s}^2}{2} + \cancel{\frac{ks^2}{2}}\ \frac{k_{eff}\, s^2}{2} \qquad k_{eff} = 4k$$

$$= \boxed{\frac{m\dot{s}^2}{2} + 2ks^2}$$

can also get from

$$PE_{tot} = \frac{ks^2}{2} + \frac{3ks^2}{2} = 2ks^2.$$

1(d) Conservation of energy $\Rightarrow \dfrac{dE_{tot}}{dt} = 0$.

$$\Rightarrow \quad 0 = \frac{d}{dt}\left(\frac{m\dot{s}^2}{2} + 2ks^2\right)$$

$$= \frac{m}{2} 2\dot{s}\ddot{s} + 4ks\dot{s}$$

$$= \dot{s}\left[m\ddot{s} + 4ks\right]$$

But $\dot{s}(t) \neq 0$ for some t.

$$\Rightarrow \quad 0 = m\ddot{s} + 4ks$$

$$\Rightarrow \quad 0 = \ddot{s} + \left(\frac{4k}{m}\right)s$$

$$\Rightarrow \quad \boxed{0 = \ddot{s} + \omega_0^2 s}$$

1(e) $F_{damp} = -b\dot{s}$

EOM: Newton's 2nd law:

$$m\ddot{s} = F_{spring} + F_{damping}$$

$$= -ks - 3ks - b\dot{s}$$

$$\Rightarrow \quad \ddot{s} + \underbrace{\left(\frac{b}{m}\right)}_{2\gamma}\dot{s} + \left(\frac{4k}{m}\right)s = 0$$

$$\Rightarrow \quad \boxed{\ddot{s} + (2\gamma)\dot{s} + \omega_0^2 s = 0.}$$

$\gamma = \dfrac{b}{2m}$

(f) $\mathbb{C}$ version of EOM:

$S \to \tilde{S}$
$\mathbb{R} \quad \mathbb{C}$.

$$\ddot{\tilde{S}} + 2\gamma\dot{\tilde{S}} + \omega_0^2 \tilde{S} = 0$$

Given: $\tilde{S}(t) = Ae^{i\omega t}$

plug in $\Rightarrow$ $-\omega^2 + 2i\gamma\omega + \omega_0^2 = 0$

$$\Rightarrow \omega_\pm = \frac{-2i\gamma \pm \sqrt{-4\gamma^2 + 4\omega_0^2}}{-2}$$

$$\Rightarrow \boxed{\omega_\pm = i\gamma \pm \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\tilde{\omega}}}$$

So:
$$\tilde{S}(t) = Ae^{-\gamma t}e^{i\tilde{\omega}t} + Be^{-\gamma t}e^{-i\tilde{\omega}t}$$
$$= \boxed{\left[Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right]e^{-\gamma t}}$$

* Block comes to rest fastest w/o oscillating when critically damped.

This occurs when $\tilde{\omega} = 0 \Rightarrow \boxed{\omega_0 = \gamma}$

(g) When underdamped, $\tilde{\omega}$ is real so extracting the real solution we get:

$$Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t} \longrightarrow C\cos(\tilde{\omega}t - \phi)$$

Amplitude

$$\Rightarrow \boxed{S(t) = Ce^{-\gamma t}\cos(\tilde{\omega}t - \phi)}$$

$\Rightarrow$ Amplitude $(t) = Ce^{-\gamma t}$

$\Rightarrow$ Maximum heat energy that can be delivered aft $t = \frac{2}{\gamma}$ is:

$$\Delta E = \frac{1}{2} k_{eff} \left(\text{Amplitude}\right)^2 \left(t = \tfrac{2}{\gamma}\right) - E(t=0) \qquad k_{eff} = 4k$$

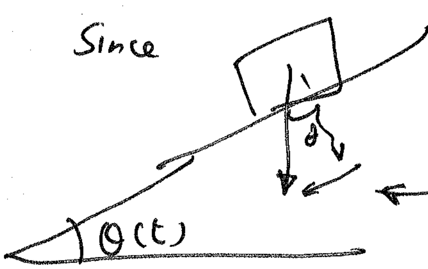
$$= \frac{4k}{2}\left[c^2 e^{-2\gamma t}\right] - \frac{4k}{2}c^2$$

$$= 2kc^2\left[e^{-2\gamma\left(\frac{2}{\gamma}\right)} - 1\right] = 2kc^2\left[e^{-4} - 1\right] < 0.$$

$\therefore$ maximal Heat delivered is $\boxed{2kc^2\left[1 - e^{-4}\right]}$

$\boxed{1(h)}$ This becomes a forced simple harmonic oscillator.

Since



$\leftarrow$ This force is $mg\sin\theta$

$$\simeq mg \sin\left(\theta_0 \cos(\omega t)\right)$$

$$F(t) \approx \left(mg\,\theta_0\right)\cos(\omega t) \quad \text{since } |\theta_0| \ll 1.$$

If $\frac{2\pi}{\omega} = 2$ yrs and $\frac{2\pi}{\omega_0} = 1$ sec

$\Rightarrow$ $\underline{\omega \ll \omega_0}$. $\Rightarrow$ On a short time scale, looks like a constant force. but on a longer time scale, can see the effect of θ oscillating w/ ω.

$$\Rightarrow \boxed{T_{block} \approx \frac{2\pi}{\omega_0} = 1 \text{ sec.}}$$

P2.

(a) Consider $g(z)$ to be any one-variable (z) function.

Then let $z = x \pm vt$.

Check if $g(x \pm vt)$ satisfies wave eq'n.

Check: LHS:

$$\frac{\partial^2 g(x \pm vt)}{\partial t^2} = \frac{\partial}{\partial t}\left[\underbrace{\frac{dg}{dz}}_{= g'(z)} \cdot \frac{\partial z}{\partial t}\right]$$

$$= (\pm v)\frac{\partial}{\partial t}\left[g'(z)\right]$$

$$= (\pm v)\left[g''(z)\underbrace{\frac{\partial z}{\partial t}}_{= (\pm v)}\right]$$

$$= v^2 g''(z).$$

RHS: $v^2 \dfrac{\partial^2 g(x \pm vt)}{\partial x^2}$

$$= v^2 \frac{\partial}{\partial x}\left[\frac{dg}{dz} \cdot \underbrace{\frac{\partial z}{\partial x}}_{= 1}\right] = v^2 \frac{\partial}{\partial x} g'(z)$$

$$= v^2 g''(z)$$

$\Rightarrow$ LHS = RHS

$\therefore$ $g(x+vt)$ & $g(x-vt)$ solutions to wave eq'n.

$$\Rightarrow \boxed{f(x,t) = g(x-vt) + g(x+vt)} \text{ sol'n to wave eq'n.}$$

1(b)

$$\boxed{\tilde{f}(x,t) = \tilde{A}\, e^{i(kx - \omega t)}} \quad \text{where} \quad \boxed{\omega = \frac{kV}{2}}$$

↳ corresponding wave eqn:

$$\boxed{\frac{\partial^2 \tilde{f}}{\partial t^2} = \left(\frac{V}{2}\right)^2 \frac{\partial^2 \tilde{f}}{\partial x^2}}$$

1(c)

Incident: $y_i(x,t) = A_i\, e^{i(k_1 x - \omega t)}$

Reflected: $y_r(x,t) = A_r\, e^{i(-k_1 x - \omega t)}$

Transmitted: $y_t(x,t) = A_t\, e^{i(k_2 x - \omega t)}$

$$\boxed{k_1 = \frac{2\pi}{\lambda_A} \qquad k_2 = \frac{2\pi}{\lambda_B}}$$

$\lambda_A \equiv$ wavelength in string A

$\lambda_B \equiv$ wavelength in string B.

1(d)

Resultant wave in A: $y_1(x,t) = y_i + y_r$

$$= [A_i\, e^{ik_1 x} + A_r\, e^{-ik_1 x}]\, e^{-i\omega t}$$

Resultant wave in B: $y_2(x,t) = y_t(x,t)$

$$= A_t\, e^{i(k_2 x - \omega t)}$$

[(e)] 2 Boundary conditions (BCs)

BC1 : 2 strings are joined at $x=0$

$\Rightarrow \quad y_1(0,t) = y_2(0,t)$

$$\Rightarrow \boxed{A_i + A_r = A_t}$$

BC2 : No kinks (smooth joint) at the joint ;

$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0}$$

$$\Rightarrow \boxed{k_1(A_i - A_r) = k_2 A_t}$$

[(f)] If density of B >> density of A ;

then transmitted wave disappears. $\Rightarrow A_t \to 0$.

Then BC1 gives $A_i + A_r \simeq 0 \Rightarrow \boxed{A_i = -A_r}$

P3

(a)

2 Guesses

(1) $y_1(x,t) = A\cos(kx)\,e^{-i\omega t}$

(2) $y_2(x,t) = B\sin(kx)\,e^{-i\omega t}$

2 BCs: $y(0,t)=0$; $y(L,t)=0$.

For Guess (1): $y_1(0,t)=0 \Rightarrow A=0. \Rightarrow y_1(x,t)=0$

For guess (2): $y_2(0,t)=0$ ✓ automatically satisfied since $\sin(0)=0$.

and, $y_2(x=L,t)=0$
$= B\sin(kL)\,e^{-i\omega t}$

$\Rightarrow kL = n\pi \quad n=1,2,3\ldots$

$\rightarrow \boxed{k_n = \frac{n\pi}{L}}$ $\Rightarrow \boxed{\omega_n = k_n v} = \frac{n\pi v}{L}$

∴ ~~Gen~~ $y_n(x,t) = y_{1,n}(x,t) + y_{2,n}(x,t)$
$= B_n\sin(k_n x)\,e^{-i\omega_n t}$ is sol'n.

To extract real ~~part of answer~~ solution

(using procedure discussed in class, expand $e^{-i\omega_n t}$)

$$\Rightarrow \boxed{y_n(x,t) = \left\{A_n\cos\left(\frac{n\pi vt}{L}\right) + B_n\sin\left(\frac{n\pi vt}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right)}$$

indeed.

(b) $\boxed{y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)}$ (pg 15)

general solution.

(c) Can just "read off" the coefficients from $g(x)$ & $h(x)$.

so: $g(x) = y(x=0,t) = \sum_{n=1}^{\infty} y_n(0,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

$\parallel$

$\frac{L}{4}\sin\left(\frac{4\pi x}{L}\right) - \frac{L}{3}\sin\left(\frac{18\pi x}{L}\right)$ match.

$\Rightarrow$ $\boxed{A_4 = \frac{L}{4}, \quad A_{18} = -\frac{L}{3}, \quad A_m = 0 \text{ for } m \neq 4, 18}$

$h(x) = \left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left.\frac{\partial y_n}{\partial t}\right|_{t=0}$

$= \sum_{n=1}^{\infty} \underbrace{(\omega_n B_n)}_{= C_n} \sin\left(\frac{n\pi x}{L}\right)$

$\parallel$

$\frac{V_0}{5}\sin\left(\frac{16\pi x}{L}\right) + \frac{V_0}{2}\sin\left(\frac{2\pi x}{L}\right)$

$\Rightarrow C_{16} = \frac{V_0}{5} \Rightarrow B_{16} = \frac{C_{16}}{\omega_{16}} = \frac{V_0}{5}\frac{L}{16\pi V}$

$= \boxed{\frac{V_0 L}{80\pi V}}$

$\Rightarrow C_2 = \frac{V_0}{2} \Rightarrow B_2 = \frac{C_2}{\omega_2} = \frac{V_0}{2} \cdot \frac{L}{2\pi V} = \boxed{\frac{V_0 L}{4\pi V}}$

$$\therefore \quad \boxed{B_2 = \frac{V_0 L}{4\pi V}, \quad B_{16} = \frac{V_0 L}{80\pi V}, \quad B_m = 0 \text{ for } m \neq 2, 16.}$$

~~$\therefore$ For $t>0$ $y(x,t) = [A_4 \cos(\omega_4 t)$~~

$\therefore$

$$\boxed{\begin{aligned}&\text{For } t>0: \\ &y(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right\} \sin\left(\frac{n\pi x}{L}\right) \\ &\text{with the } A_n \text{ \& } B_n \text{ determined above.}\end{aligned}}$$

$\boxed{1(d)}$

$$g(x) = \frac{L}{4} \sin\left(\frac{8\pi x}{2L}\right) - \frac{L}{3} \sin\left(\frac{18\pi x}{L}\right)$$

$$= \frac{L}{4} \sin\left(\frac{8\pi x}{2(L/2)\cdot 2}\right) - \frac{L}{3} \sin\left(\frac{18\pi x}{2\cdot(L/2)}\right)$$

$$= \frac{L}{4} \sin\left(\frac{2\pi x}{(L/2)}\right) - \frac{L}{3} \sin\left(\frac{9\pi x}{(L/2)}\right)$$

$$\Rightarrow \quad A_2 = L/4 \qquad A_9 = -L/3 \qquad A_m = 0 \text{ for } m \neq 2, 9$$

$$h(x) = \frac{V_0}{5} \sin\left(\frac{8\pi x}{L/2}\right) + \frac{V_0}{2} \sin\left(\frac{\pi x}{(L/2)}\right)$$

$$\Rightarrow \quad B_8 = \frac{V_0}{5\omega_8} \qquad B_1 = \frac{V_0}{2\omega_1} \qquad B_m = 0 \text{ for } m \neq 1, 8.$$