Physics III (MITES '08)



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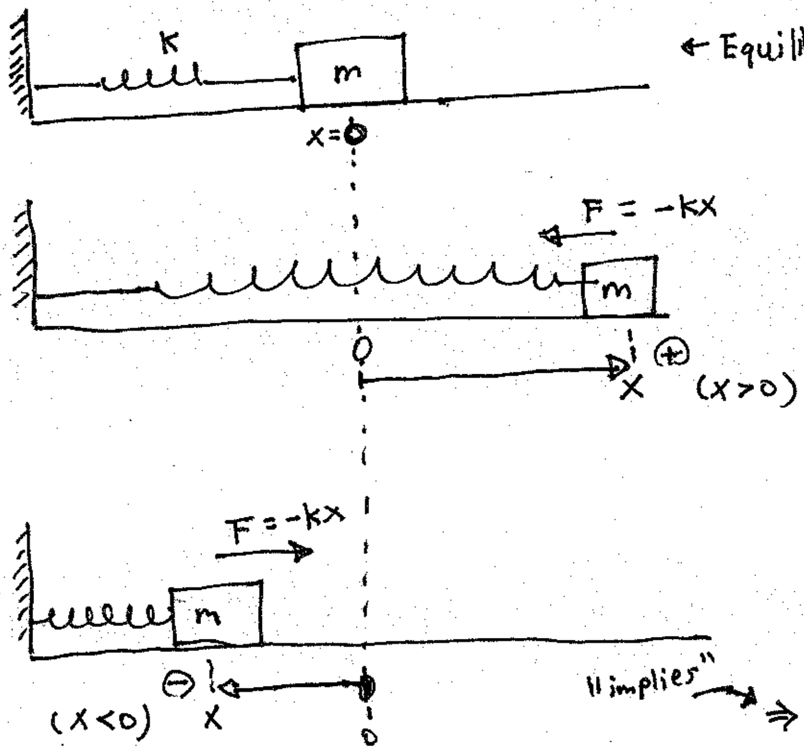
# 1. Oscillations of a particle

**Example**: A block of mass m attached to a Hookian spring (w/ spring constant k) (Frictionless floor)




← Equilibrium position (i.e. No force is acting on the block since spring is at its rest length)

- Force on the block due to displacement $x$ is $F=-kx$.
  ↑ (can be ⊕ or ⊖).
- Notice that dimension of spring constant k is $\frac{\text{Newton}}{\text{meter}}$.

so writing Newton's 2nd law:

$F = -kx$ "implies" $\Rightarrow ma = -kx$ — "defined as" $a \equiv$ acceleration

But, $a = \frac{dV}{dt}$ — $V \equiv$ velocity.

$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$ ← since $V = \frac{dx}{dt}$

$= \frac{d^2x}{dt^2}$

so, Newton's 2nd law gives us:

$$m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0. \quad \text{(rearranging)}$$

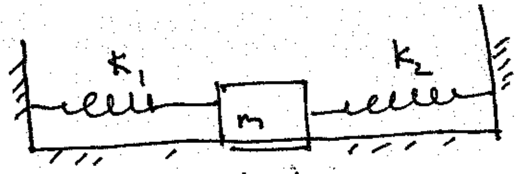
Eq'n (1) → $$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$ ← Further rearranging (i.e. dividing by m) gives us this differential equation

- Differential Equation is an equation with an unknown (in this case, "$x$") and its derivatives (in this case, "$\frac{d^2x}{dt^2}$"). Basically, Newton's 2nd law just gave us an equation describing how the block moves, and by solving the boxed equation above (solve for $x(t)$), we can find the motion of the block as a function of time (since $x(t)$ is the position of block at time $t$).

For the reason stated at the bottom of previous page,

the equation: $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ is called the "Equation of motion" (EOM).

**Another example:** Qu: Derive the equation of motion (EOM) of the following system:



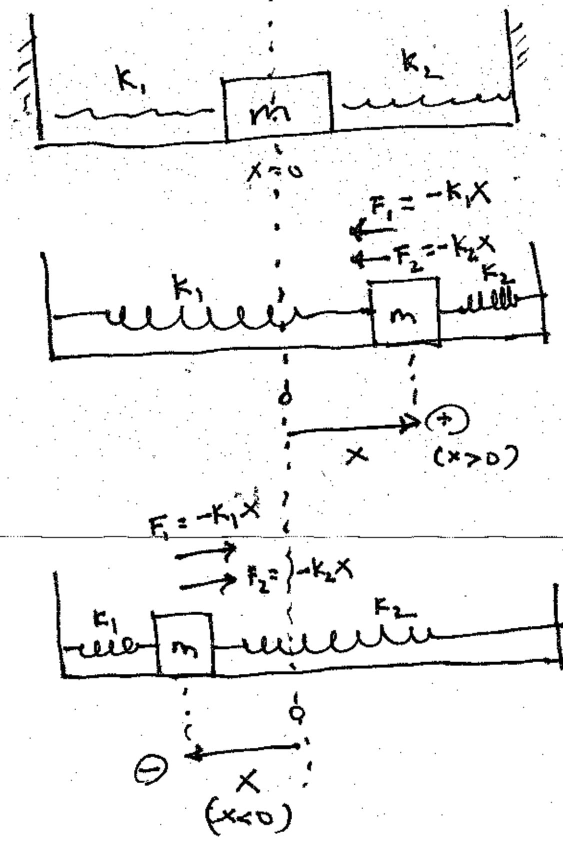

No friction.

$k_1, k_2$ = spring constants.

$m \equiv$ mass of block.

Assume the springs obey Hooke's law.

sol'n:




Equilibrium position (No force on the block by either spring)

← Net (total) force on block

$$F_{net} = F_1 + F_2$$
$$= -k_1x - k_2x$$
$$= -(k_1+k_2)x$$

$\therefore\ ma = -(k_1+k_2)x$ ← Newton's 2nd law

"Therefore"

But we know that:

$$a = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\therefore\ m\frac{d^2x}{dt^2} = -(k_1+k_2)x$$

$$\boxed{\text{Eq'n (2)}} \rightarrow\ \Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{(k_1+k_2)}{m}x = 0}$$

↑ EOM.

- Both EOMs that we just derived have the same form (i.e. they "look" the same like this:

$$\boxed{\frac{d^2x}{dt^2} + (\text{some } \#)\,x = 0} \leftarrow \boxed{\text{Eq'n (3)}}$$

What is this "some #"?

- To get a sense of what that "some #" is, look at the dimension (units) of that quantity:

In Eq'n (1) (on pg 1) : "some #" $= k/m$.

<u>Dimension of k/m</u> : $\left[\frac{k}{m}\right] = \frac{[k]}{[m]}$

$$= \frac{\left(\frac{\text{Newton}}{\text{meter}}\right)}{(\text{kg})}$$

(Notation: $[\cdots] =$ dimension (units) of $\cdots$

e.g. $[V] = \text{meter}/s$

velocity   ~~meter~~ s = seconds)

Since
$F = ma$
$\Rightarrow \text{Newton} = kg \cdot m/s^2$ $\longrightarrow$

$$= \frac{\cancel{kg} \cdot \cancel{meter}}{\cancel{meter} \cdot s^2} \frac{1}{\cancel{kg}} = \frac{1}{s^2}$$

- similarly, In eq'n (2) (on pg 2), $\underbrace{\left[\frac{k_1 + k_2}{m}\right]}_{= \text{"some \#"}} = \frac{1}{s^2}$.

- Hence, that "#" in both eq'n (1) & (2) have the <u>same</u> unit. $\left(\frac{1}{\text{time}^2}\right)$.
- Also, both $\frac{k}{m}$ and $\frac{k_1 + k_2}{m}$ are <u>positive</u>.
- Motivated by this, we let (define) $\omega^2 \equiv k/m$ (in eq'n (1))

("defined as")

and $\omega^2 \equiv \frac{k_1 + k_2}{m}$ (in eq'n (2)).

(i.e. $\boxed{\omega^2 \equiv \text{"some \#" in eq'n (3)}}$) $\Rightarrow$ Eq'n (3) now reads : $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$

Eq'n (4)

$\Rightarrow$ $[\omega] = \frac{1}{\text{time}}$ (e.g. $\frac{1}{\text{sec}}$, $\frac{1}{\text{hr}}$, $\frac{1}{\text{min}}$, $\frac{1}{\text{light year}}$, etc.).

("implies")

dimension of <u>frequency</u>

More specifically, $\frac{1}{\text{sec}} \equiv \text{Hz}$ (Hertz)

- We will see later that $\omega$ is the <u>angular frequency</u> of motion.

- Is it possible that any object (could be elephant, electron, volume of air in your lung, electrical current in AC circuit) that oscillates ~~always~~ has an "equation of motion" that looks like Eq'n (4)?

We can answer this question by solving for the unknown function $x(t)$ in the differential eq'n (4), and then seeing if $x(t)$ is an "oscillating" function. If it is, then indeed we can say that any object whose EOM looks like Eq'n (4), indeed ~~always~~ oscillates.

- But before solving Eq'n (4), let's derive the EOMs of the 2 previous examples using a different method (i.e. through conservation of energy, instead of Newton's 2nd law).

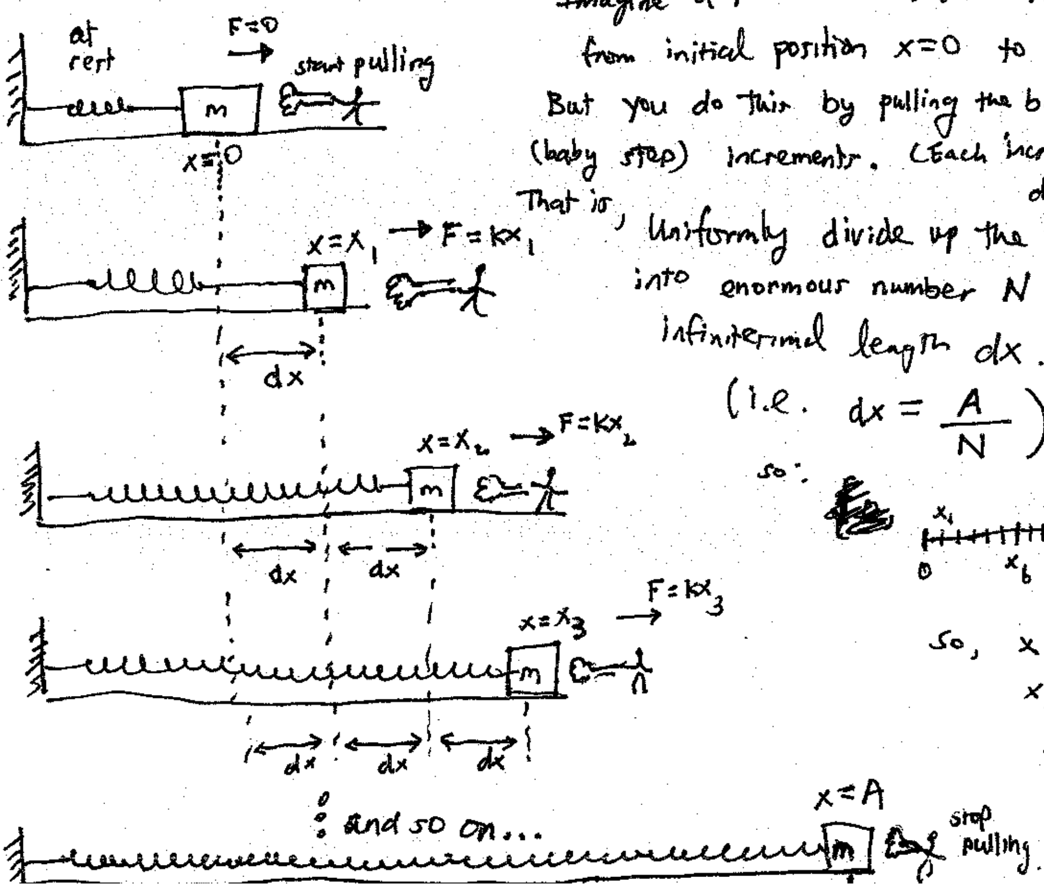
EX: Going back to the example on (pg 1), let's derive the EOM (eq'n (1)) using conservation of energy.

As a warm up, let's figure out what the total energy of the system (= block + spring) is using the following procedure:

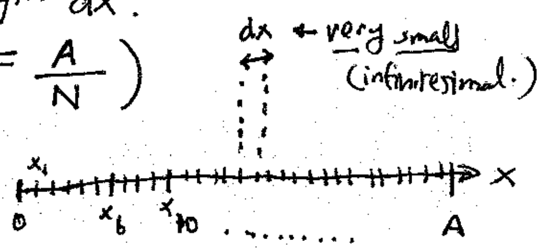
Imagine a process in which you pull the block from initial position $x=0$ to final position $x=A$. But you do this by pulling the block in infinitesimal (baby step) increments. (Each incremental displacement is $dx$)

That is, uniformly divide up the interval $[0, A]$ into enormous number $N$ of segments, each with infinitesimal length $dx$.

$$\left(\text{i.e. } dx = \frac{A}{N}\right)$$




so:




So, $x_1 = 0 + dx$

$x_2 = 0 + 2dx = x_1 + dx$

$x_3 = 0 + 3dx = x_2 + dx$

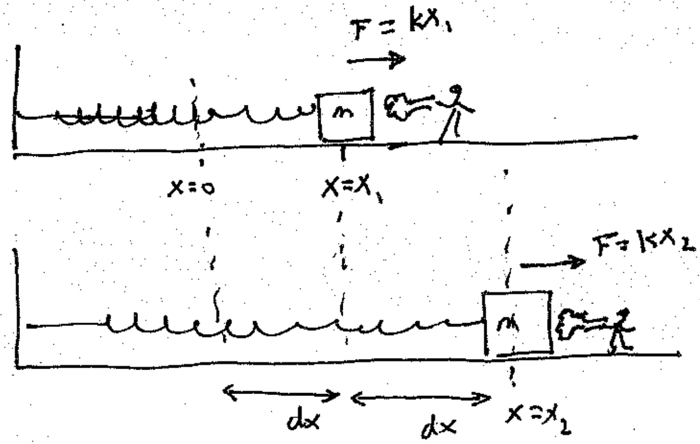
$\vdots$ and so on.

So, the procedure for getting the block from $x=0$ to $x=A$ is:

- First pull the block from $x=0$ to $x_1 = 0+dx$,
- Next, pull the block from $x_1 = dx$ to $x_2 = x_1 + dx$,

⋮ and so on until you reach $x = A$.

- Since the force exerted by the spring on the block gets stronger as you're pulling it farther away from the equilibrium position ($x=0$), your hand has to exert the same amount (as the spring) of increasing force on the block to ~~contect~~ counteract the force of spring. What exactly is the force that your hand has to exert in going from $x_1$ to $x_2$? (see diagram on pg 4)




(Note: dx looks longer than it should for visual aid. But it's supposed to be infinitesimal.)

- F increases from $kx_1$ to $kx_2$ while you're moving it from $x_1$ to $x_2$ (force exerted by you on the block)

But, since $x_2 - x_1 = dx$ ← very small,

$$\Rightarrow \quad kx_2 - kx_1 = k\,dx$$

And since dx is incredibly small, so is $k\,dx$.

Since the force change is so incredibly small, when moving from $x_1$ to $x_2$, we can, to a very good approximation, think of F remaining essentially constant (unchanged) over the interval $[x_1, x_2]$. And the work done by your hand, in moving the block from $x_1$ to $x_2$ is:

$$W_{1\to 2} = (\text{Force at } x_1)\cdot \overbrace{dx}^{= x_2 - x_1}$$

$$= \underbrace{kx_1\,dx}.$$

↑ because dx appears here as well, $W_{1\to2}$ is also an infinitesimal quantity. So we write $dW_{1\to2}$ instead of writing $W_{1\to2}$. (To emphasize that it's small.)

So, rewrite as: $\boxed{dW_{1\to2} = kx_1\,dx}$

- Going back to the diagram on (Pg 4), we can apply the logic involved in computing $dW_{1\to 2}$ to ~~other~~ the other displacement increments.
  (e.g. to $[x_3, x_4]$, $[x_4, x_5]$, etc.)

The total work done by your hand in moving the block from $x=0$ to $x=L$ is:

$$W_{total} = dW_{x=0\to x_1} + dW_{1\to 2} + dW_{2\to 3} + \cdots$$

$$= \sum_{x=0}^{x=A} dW$$

where: $dW_{1\to 2} = (kx_1)dx$
$dW_{2\to 3} = (kx_2)dx$
$\vdots$ and so on.

$$= \sum_{x=0}^{x=A} (kx)dx$$

$$\equiv \int_{x=0}^{A} kx\,dx = \frac{kx^2}{2}\Bigg|_{x=0}^{A} = \boxed{\frac{kA^2}{2}}.$$

by definition, this is what integration is !!

$\therefore$ The total amount of work done by your hand in moving the block from the equilibrium position $(x=0)$ to final position $(x=A)$

("Therefore") is $W_{total} = \frac{kA^2}{2}$.

And, by the Work-energy theorem (i.e. "The work you do on a system is the total amt. of energy of that system"), you have inserted total energy of $\frac{kA^2}{2}$ to the system (= block + spring).

When you release the block at $x=A$, the total amount of energy of the system will forever be $\boxed{E_{tot} = W_{tot} = \frac{kA^2}{2}}$

(Pg 7)

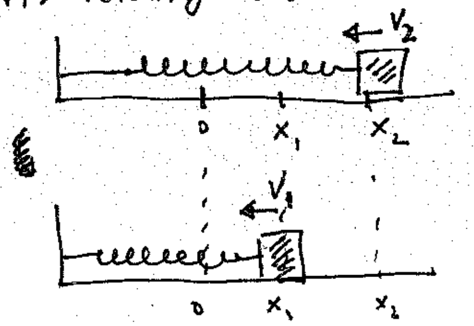
• But we're not done yet! We know what the total energy is: ( $KA^2/2$ )
but what is the kinetic & potential energy as a function of the position of the block $x$ ?

To answer this, go back to Newton's $2^{nd}$ law: but now, consider the block moving from $x_2$ to $x_1$ $(x_1 - x_2 = dx)$ ← the small interval again. [Notice that here, $dx<0$ since $x_1<x_2$.]

Newton's $2^{nd}$ law tells us that:

$$\boxed{\begin{array}{ccc} \text{impulse} & = & \text{change in momentum.} \\ \| & & \| \\ F\Delta t & & mV_{final} - mV_{initial}\,. \end{array}}$$

After releasing the block:




} little later (i.e. $dt$)

Since $x_1$ is so close to $x_2$, the time taken to go from $x_2$ to $x_1$ is very small (infinitesimal). Call that time interval $dt$.

And since $x_1$ is so close to $x_2$, there's not much time for velocity to change much;

$$\left( \text{i.e.} \quad mV_1 - mV_2 = m\underbrace{(V_1 - V_2)}_{\text{small (infinitesimal)}} = m\,dV_{2\to 1} \right)$$

so, impulse = change in momentum applied to the interval $(x_1, x_2)$ reads:

$$F\,dt = m\,dV_{2\to 1}$$

$$\Rightarrow \boxed{-kx_2\,dt = m\,dV_{2\to 1}} \quad \leftarrow \text{Eq'n (5)}$$

• Now, as on (pg 6), we can apply this to all the other intervals; ~~and get~~: (there's nothing special about $x_1$ & $x_2$)

so dropping the subscripts in Eq'n (5), we write:

$$-kx\,dt = m\,dV$$

$$\Rightarrow -kx = m\frac{dV}{dt} \quad \leftarrow \text{(divide both sides by } dt\text{)}$$

$$\Rightarrow -kx = m\frac{dV}{dx}\left(\frac{dx}{dt}\right) \quad \leftarrow \left(\text{multiply by } \frac{1}{dx}\cdot dx = 1\right)$$

$$\Rightarrow -kx\,dx = m(dV)\left(\frac{dx}{dt}\right) \quad \leftarrow \text{(multiply both sides by } dx\text{)}$$

$$\Rightarrow (-kx)dx = (mV)dV \quad \leftarrow \because \frac{dx}{dt} = V.$$

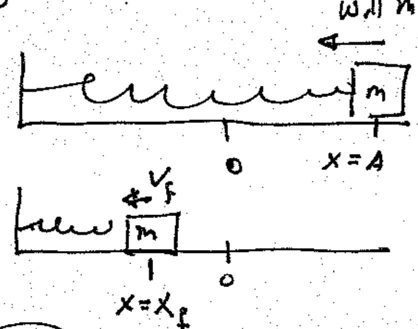
continued... (over)

Now, $\boxed{(-kx)\,dx = (mv)\,dV}$ ← Eq'n (6)

• Imagine going from $x = A$ (Initially at rest $V=0$) to some position $x = x_f$ :

Let $V_{initial} = V(x=A) = 0$.

$V_f = V(x = x_f)$.




will move this way just after released from your hand at $x=A$. (Initially, $V_i = 0$)

Then, following the same logic as on Pg 6, we break up the interval $[x_f, A]$ into many segments, each with infinitesimal length $dx$, and ~~then sum all these intervals so~~. sum eq'n (6) over all these intervals.

So:
$$\sum_{x=A}^{x_f} (-kx)\,dx = \sum_{V_i=0}^{V_f} (mv)\,dV$$

$$\Rightarrow \int_{x=A}^{x_f} (-kx)\,dx = \int_{V=0}^{V_f} (mv)\,dv$$

By definition, this is integral!

$$\Rightarrow -\frac{kx^2}{2}\Bigg|_A^{x_f} = \frac{mv^2}{2}\Bigg|_0^{V_f}$$

$$\Rightarrow -\frac{kx_f^2}{2} + \frac{kA^2}{2} = \frac{mV_f^2}{2}$$

$$\Rightarrow \frac{kA^2}{2} = \frac{mV_f^2}{2} + \frac{kx_f^2}{2}$$

Now, we can choose $x_f$ (and thus $V_f = V(x=x_f)$) to be any position we want. So $x_f$ (and thus $V_f$) is a variable. Motivated by this, we drop the subscript "f" and write:

see Pg 6 → $E_{total} = \boxed{\frac{kA^2}{2} = \frac{mV^2}{2} + \frac{kx^2}{2}}$

$\frac{mv^2}{2}$ = kinetic energy of block

$\frac{kx^2}{2} \equiv$ potential energy of spring.

· ~~We~~ We have just derived conservation of energy for the block-spring problem.

· Total Energy is conserved since: $\dfrac{dE_{total}}{dt} = \dfrac{d}{dt}\left(\dfrac{kA^2}{2}\right) = 0.$

every term here is constant. (Not a function of time)

(i.e. when we say that the total energy is conserved, we mean that it's the same value at all times.

$\Rightarrow \boxed{\dfrac{dE_{total}}{dt} = 0}$ )

so; we're finally in position to derive the EOM of block-spring system (on Pg 1) using ~~the energy of the system~~ conservation of energy.

Note that:

$$0 = \frac{dE_{total}}{dt} = \frac{d}{dt}\left[\frac{kx^2}{2} + \frac{mv^2}{2}\right]$$

( k, m constants (i.e. Not function of time)
$x = x(t)$
$v = v(t)$ )

$$= \frac{k}{2}\frac{d(x^2)}{dt} + \frac{m}{2}\frac{d}{dt}(v^2)$$

$$= \frac{k}{2}\,2x\frac{dx}{dt} + \frac{m}{2}\,2v\frac{dv}{dt}$$

$$= kx\underbrace{\left(\frac{dx}{dt}\right)}_{=v} + mv\underbrace{\left(\frac{dv}{dt}\right)}_{=\frac{d^2x}{dt^2}}$$

$$= kxv + mv\frac{d^2x}{dt^2}$$

$$= v\left[kx + m\frac{d^2x}{dt^2}\right]$$

Divide both sides by $v(t)$ to get:

$$0 = kx + m\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega^2}x = 0 \quad \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

We just derived EOM (same as Pg 1) using conservation of energy.