June 25, 08 [Wed.]

- Summary of what we've done so far:

  - Derived EOM (Method 1: Newton's 2^{nd} law. Method 2: Conservation of energy.)

  - Noticed that 2 different systems we know would oscillate, have the same form of eqn: (pg 2):

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

where ω has unit of frequency ($1/\text{time}$)

Eqn (7)

- Going back to (pg 4) (read the question posed at the top there),

can we show that

① ~~Eq~~ Any object (could be elephant, electron, volume of air in your lung, electrical current in AC circuit etc.) ~~that obeys~~ whose EOM is the same form as eqn (7) oscillates?

② Do objects that oscillate have to obey Eqn (7) and no other EOM?

- ② is ~~...~~ harder to prove, so we will first show that ① is true.

~~...~~

- To do so, we simply need to solve for $x(t)$ in eqn (7) and see if it's a function that oscillates.

~~...~~

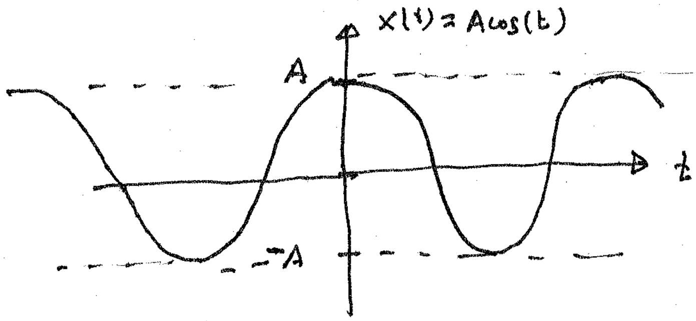
over →

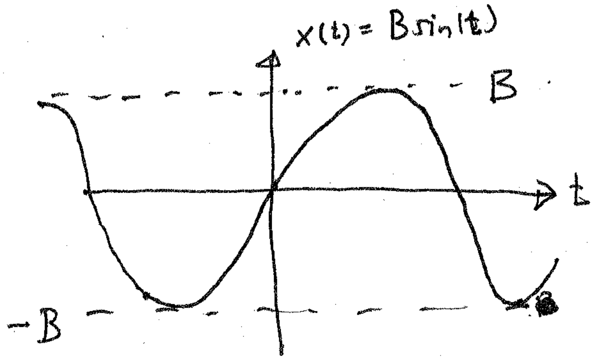
Qu: Solve for $x(t)$ in the differential eq'n: $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$

Sol'n: Look at "Math Appendix A: Introduction to Differential Eqn's: Part I" handout ~~for~~ the 2 general methods of solving differential eqn's.

Here's we'll use [method 1: Guess the solution (i.e. guess what $x(t)$ is) then check by plugging into the eq'n.].

- We know that cosine & sine are 2 well known oscillatory functions. so why don't we try $\cdot\ x(t) = A\cos(t)$ and $x(t) = B\sin(t)$





- First, with our guess $x(t) = A\cos(t)$; (A is the amplitude) which can be anything.

$$\frac{dx}{dt} = -A\sin(t)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -A\cos(t)$$

So, plugging into the differential eq'n (Boxed eqn at the top of this pg.), we get:

$$\frac{d^2x}{dt^2} + \omega^2 x$$

$$= -A\cos(t) + \omega^2 A\cos(t)$$

$$= [A\cos(t)][\omega^2 - 1]$$

← Only equal to zero (meaning "zero for all values of t") if and only if $\omega^2 - 1 = 0$

$$\neq 0 \quad (!)$$

So $x(t) = A\cos(t)$ is NOT a sol'n of the differential eq'n.
(You'll find that $x(t) = B\sin(t)$ is also NOT a solution).

over →

But in fact, there is a much bigger problem with the last line of the eq'n on previous pg (Pg 11); $[A\cos(t)][\omega^2-1]$.

- The problem is that $\cos(t)$ and ω^2-1 both do not make any sense!
- To see this, notice that $[\omega^2] = \frac{1}{\text{time}^2}$ and "1" has no units.

"dimension of"
(see Pg 3)

So we cannot subtract 1 from ω^2 in ω^2-1.
(This is like saying "3 seconds plus 2 apples" [= 5 (what units?!!)]
You can only add 2 ~~things~~ quantities that have the same physical units.
This is one of the many things that sets physics apart from math.

- For the same reason, $\cos(t)$ doesn't make sense.

$[t] \equiv$ time (e.g. seconds, Hrs, etc.)

but what unit does $\cos(t)$ have?
(e.g. $\cos(2) \approx -0.42$. so is it true that $\cos(2\text{ apples}) = -0.42$ (units?!!))

"Approximately"

Again, doesn't make sense.)

In fact, from this reasoning, it follows that:

$\cos(\ldots)$ only makes sense
If and only if $\ldots$ is dimensionless
(Has no units)
(Is just a number.)

• But cosine & sine are still good guesses (since they oscillate) (periodic functions) for $x(t)$, but we just need to make sure that the argument of cos & sin is <u>dimensionless</u> (No units)

To do so, let α = some unknown #, with $[\alpha] = \frac{1}{\text{time}}$.

And let $\underline{x(t) = A\cos(\alpha t)}$

↑ our guess

[Notice that $[\alpha t] = [\alpha][t] = \frac{1}{\cancel{\text{time}}} \cdot \cancel{\text{time}} = 1$ ← dimensionless (Just as we want.)]

Plugging this $x(t)$ into eq'n (7) on pg10 gives us:

$$\frac{d^2x}{dt^2} + \omega^2 x = -A\alpha^2\cos(\alpha t) + \omega^2 A\cos(\alpha t)$$

$$= \left(A\cos(\alpha t)\right)\left(-\alpha^2 + \omega^2\right)$$

← Since:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left[-\alpha A\sin(\alpha t)\right] = -\alpha A \frac{d}{dt}\sin(\alpha t) = -\alpha^2 A\cos(\alpha t)$$

so, $x(t) = A\cos(\alpha t)$ is indeed a solution to our differential eq'n [eq'n (7) on pg10] if and only if:

$$0 = [A\cos(\alpha t)](\omega^2 - \alpha^2)$$

And, this can only hold (at <u>all</u> values of t) if and only if $\omega^2 - \alpha^2 = 0$

$$\Rightarrow \boxed{\alpha = \pm\omega}$$

So, $\boxed{x(t) = A\cos(\omega t) \text{ is a solution}}$ [← Note $\cos(-(\cdots)) = \cos(+(\cdots))$]

• And what about $x(t) = A\sin(\alpha t)$?

Again, plug in this $x(t)$ into our differential eq'n to find out:

$$\frac{dx}{dt} = \alpha A\cos(\alpha t) \Rightarrow \frac{d^2x}{dt^2} = \frac{d}{dt}\left[\alpha A\cos(\alpha t)\right]$$
$$= -\alpha^2 A\sin(\alpha t)$$
$$= -\alpha^2 x(t)$$

So:
$$\frac{d^2x}{dt^2} + \omega^2 x = -\alpha^2 x(t) + \omega^2 x(t)$$
$$= (\omega^2 - \alpha^2)\, x(t)$$

• Now, again as before, we want to make $\frac{d^2x}{dt^2} + \omega^2 x = 0$.

(At all values of time t).

This can only happen if and only if $\omega^2 = \alpha^2 \Rightarrow \boxed{\alpha = \pm\omega}$.

Thus, $\boxed{x(t) = A\sin(\omega t)}$ is also a solution to the differential eq'n $\frac{d^2x}{dt^2} + \omega^2 x = 0$ as well.

Notice that $\sin(-(\cdots)) = -\sin((\cdots))$

so $A\sin(-\omega t) = -A\sin(+\omega t)$.

But since we're free to choose "A" to be any number (incl. $\ominus$ #'s), we can "absorb" the $\ominus$ sign into this free coefficient "A" and just write

$x(t) = A\sin(\omega t)$ [← takes care of $A\sin(-\omega t)$] case as well.

• So, just as $x^2 + 3x + 2 = 0$ has 2 solutions ($x = -1$ and $x = -2$),

$\frac{d^2x}{dt^2} + \omega^2 x = 0$ has 2 solutions $\left(\begin{array}{l} x(t) = A\cos(\omega t) \\ x(t) = B\sin(\omega t) \end{array}\right)$

[where A and B are any arbitrary numbers.]

• But how do we know there aren't any other solutions to $\frac{d^2x}{dt^2} + \omega^2 x = 0$?

- In fact, there are (a lot) more solutions to the EOM: $\frac{d^2x}{dt^2} + \omega^2 x = 0$.

For example, let $x_1(t) = A\cos(\omega t)$
$x_2(t) = B\sin(\omega t)$, where A & B are arbitrary constants.

Let's say $x(t) = x_1(t) + x_2(t)$

Does this $x(t)$ satisfy $\frac{d^2x}{dt^2} + \omega^2 x = 0$?

Check: Again, plugging in $x(t)$ into $\frac{d^2x}{dt^2} + \omega^2 x$, we get:

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{d^2(x_1+x_2)}{dt^2} + \omega^2(x_1+x_2)$$

$$= \frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} + \omega^2 x_1 + \omega^2 x_2$$

$$= \underbrace{\left(\frac{d^2x_1}{dt^2} + \omega^2 x_1\right)}_{=0} + \underbrace{\left(\frac{d^2x_2}{dt^2} + \omega^2 x_2\right)}_{=0}$$

since $x_1(t) = A\cos(\omega t)$ is a solution to our differential eq'n

since $x_2(t) = B\sin(\omega t)$ is a solution to our differential eq'n.

$$= 0.$$

$\therefore$ Indeed, $\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t)}$ is a solution to our EOM.

[In fact, the 2 solutions we found before: $x(t) = A\cos(\omega t)$, $x(t) = B\sin(\omega t)$ are just special cases of solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

i.e. You get $A\cos(\omega t)$ by setting $B = 0$.
You get $B\cos(\omega t)$ by setting $A = 0$.]

* It turns out that $x(t) = A\cos(\omega t) + B\sin(\omega t)$ is the most **general** solution to the differential eq'n (EOM): $\frac{d^2x}{dt^2} + \omega^2 x = 0$. (There are no other solution.)

* But what do the coefficients A & B mean physically?

To answer this, we rewrite $x(t) = A\cos(\omega t) + B\sin(\omega t)$ in a slightly different way:

Let C be another number, and consider

$C\cos(\omega t - \phi)$ [where ϕ is yet another #, soon to be determined.]

Now, using some trigonometry tricks, we get:

$$C\cos(\omega t - \phi) = C\{\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin\phi\}$$
$$= [C\cos\phi]\cos(\omega t) + [C\sin\phi]\sin(\omega t)$$

Note: we used the trig. identity: $\cos(y - z) = \cos(y)\cos(z) + \sin(y)\sin(z)$

(and thus it follows: $\cos(y+z) = \cos(y)\cos(z) - \sin(y)\sin(z)$)

(we have set $y = \omega t$ and $z = \phi$ to do the above calculation.)

so: $C\cos(\omega t - \phi) = [C\cos\phi]\cos(\omega t) + [C\sin\phi]\sin(\omega t)$

so, if we can find "C" and "ϕ" such that:

$$\left\{ \begin{array}{ll} (1) & C\cos\phi = A \\ \text{and } (2) & C\sin\phi = B \end{array} \right\} \leftarrow \text{2 eqns w/ 2 unknowns: "}\phi\text{" "C"}$$

then we'd have

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \phi)$$

Solving (1) & (2):

$(1)^2 \Rightarrow C^2\cos^2\phi = A^2$

$(2)^2 \Rightarrow C^2\sin^2\phi = B^2$

$\therefore (1)^2 + (2)^2 \Rightarrow C^2[\underbrace{\cos^2\phi + \sin^2\phi}_{=1}] = A^2 + B^2$

$\Rightarrow \boxed{C^2 = A^2 + B^2}$

And: $\frac{(2)}{(1)} \Rightarrow \frac{\cancel{C}\sin\phi}{\cancel{C}\cos\phi} = \frac{B}{A} \Rightarrow \boxed{\tan\phi = B/A}$ i.e. $\boxed{\phi = \text{Arctan}(B/A)}$.

• If we're given A & B, then we can always find C & ϕ by the 2 eqn's we derived at the bottom of pg 16.

(Also, the other direction is true: Given C & ϕ, we can find A & B by the relations:

$A = C\cos\phi$
$B = C\sin\phi$.)

So, from now on, we'll write $\boxed{x(t) = C\cos(\omega t - \phi)}$

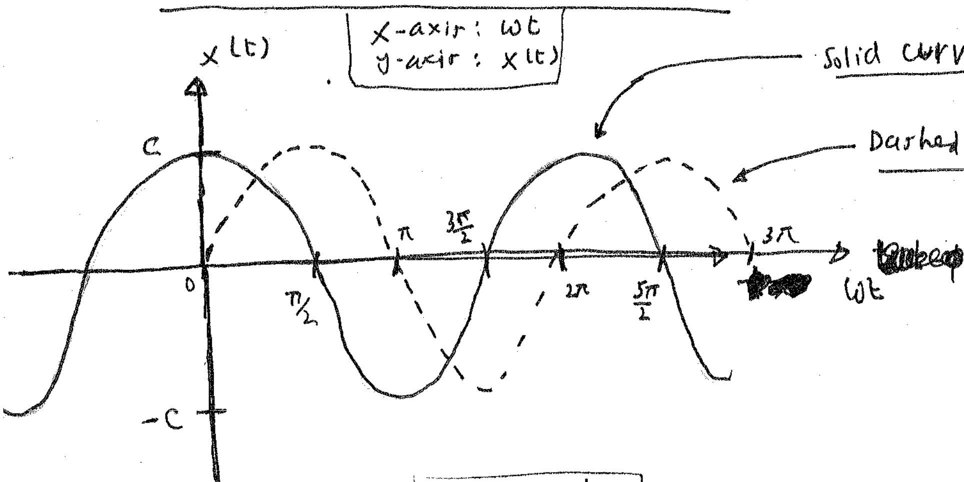
Most general soln to EOM: $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$
(solution)

• So, instead of trying ~~to~~ to answer the question: "What do the coefficients A & B mean physically?", (posed at top of pg 16)

we now try to answer the question:

Qu: "What do the #'s "C" and "ϕ" physically mean?"

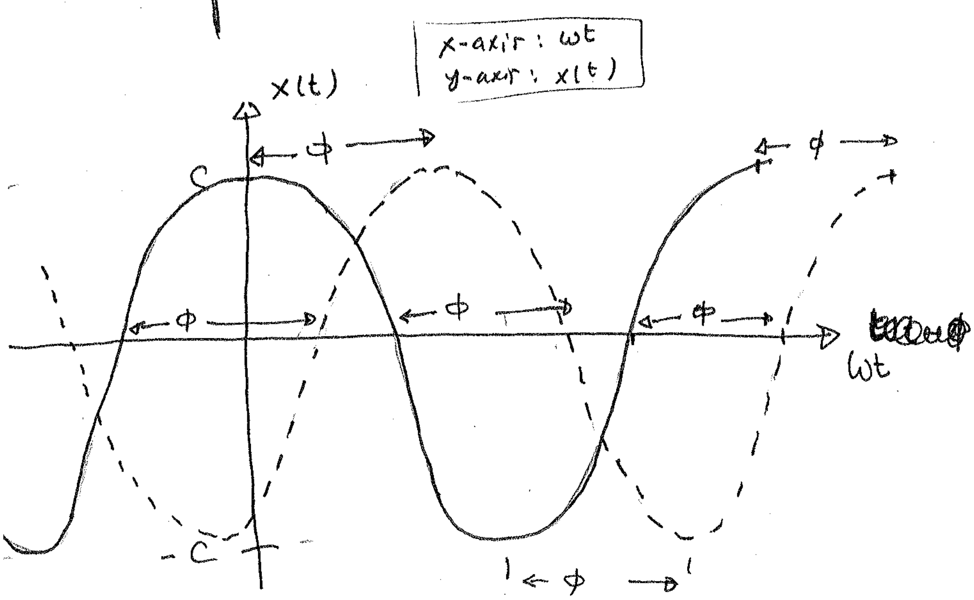
Ans: Draw (plot) $x(t) = C\cos(\omega t - \phi)$:



Solid curve: $x(t) = C\cos(\omega t)$ $[\phi = 0]$

Dashed curve: $x(t) = C\cos(\omega t - \frac{\pi}{2})$ $[\phi = \pi/2]$

(Note: In this course, all angles are measured in radians. So $\phi = \pi/2$ (radians) (i.e. 90 degrees.))



Solid curve: $x(t) = C\cos(\omega t)$

Dashed curve: $x(t) = C\cos(\omega t - \phi)$

- So, $X(t) = C\cos(\omega t - \phi)$

position.

C = Amplitude of oscillation

ω = Angular frequency. (Hz)

ϕ = "phase constant"

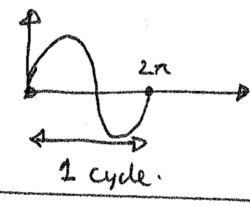
"Phase" of oscillation.

- "dimension of" $\left[\frac{2\pi}{\omega}\right]$ = seconds

$\frac{2\pi}{\omega}$ is in fact the period $\left(= \frac{1}{f}\right)$ of oscillation.

And, $\frac{\omega}{2\pi} = f$ ← frequency.

[since: If $\omega = 2\pi$ Hz, then it means the phase has been shifted by 2π radians (360°) in 1 sec. ⇒ 1 cycle /sec = 1 Hz ← frequency f.]



- THE solution to EOM: $\frac{d^2x}{dt^2} + \omega^2 x = 0$ ← $\boxed{\text{Eqn' 8}}$

is $x(t) = C\cos(\omega t - \phi)$

There are no other solutions. ↑ most general sol'n.

$x(t) = C\cos(\omega t - \phi)$ is clearly a periodic function; and since any physical system whose EOM is Eqn'(8) must have its position $x(t)$ described by $x(t) = C\cos(\omega t - \phi)$, we have just showed (proved) that any system whose EOM obeys Eqn' 8 must oscillate.

This is called "Simple Harmonic Motion". aka. (SHM)

(1) Also, on pg 15, we showed that if $x_1(t)$ & $x_2(t)$ are solutions to Eqn 8, then so is $x(t) = x_1(t) + x_2(t)$. This is the "linearity" property of SHM.

(2) Notice also that $x(t) = C\cos(\omega t - \phi)$ is periodic (in time t, not position) with period $\frac{2\pi}{\omega}$ ← (has dimension of time). This means the system looks the same (in every aspect including position, velocity, kinetic energy, potential energy etc.) if you take snapshots in every $\frac{2\pi}{\omega}$ intervals of time.

This is called "Translational symmetry" property of SHM.

over

- ~~...~~
- (1) & (2) on pg 18 are the two of these properties (3rd property being that there's some force acting on the system [e.g. system of block + spring certainly has force involved: spring exerts force on block]) that I said every object oscillating (simple Harmonically) must obey. Indeed, we have just proven that claim.

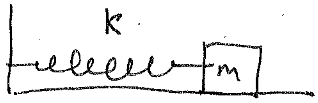
- Notice that $x(t) = C\cos(\omega t - \phi)$ has <u>2 free parameters</u>: C and ϕ
  (Amplitude) (phase constant)

  (Recall from pg 15 & pg 16 that "C & ϕ" rose out of the 2 free parameters "A" & "B".)

<u>Why is this so?</u>

(By "free parameters", we mean that the actual numerical values of "C" & "ϕ" can <u>vary</u> for the <u>same</u> physical system.)

<u>e.g.</u>



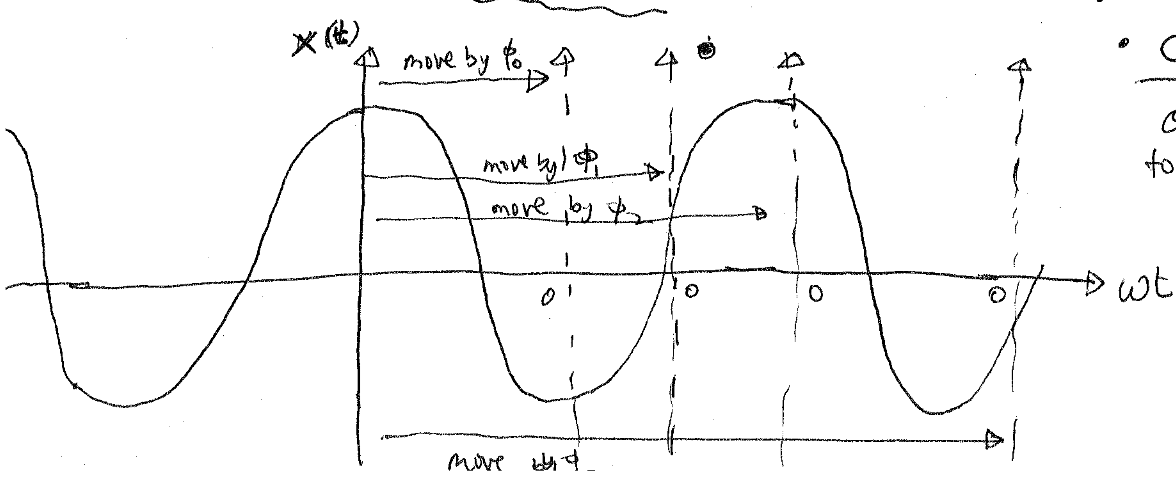
← For this physical system, the spring constant k and mass of block m are <u>fixed</u> parameter.

Thus $\omega = \sqrt{\frac{k}{m}}$ is a <u>fixed parameter</u>.

But "C" and "ϕ" are not fixed; they are <u>not</u> parameters that are given in this problem. And we found on pg 16 & 15 that C & ϕ (and equivalently, A & B) can be any arbitrary value and still have $x(t) = C\cos(\omega t - \phi)$ be a solution to the EOM. "C" & "ϕ" : <u>2 degrees of freedom</u>.

- How do we decide what values "C" and "ϕ" should be?

<u>Ans</u>: <u>Initial condition</u> of the system. fixes the 2 degrees of freedom: C & ϕ.



- C (Amplitude): You decide, at $t = 0$, by how much to pull the block away from its equilibrium position, before letting it go. (you <u>decided</u>) Not the block + spring.

- ϕ (phase constant) : You decide, where $t=0$ is.

(i.e. you decide when to start the timer on your stopwatch to begin recording the motion of block. You don't necessarily have to start the timer at the moment that you release the block.)

(see the diagram at the bottom of previous page (Pg 19) : You may choose to start the timer $t_0 = \frac{\phi_0}{\omega}$ ~~seconds~~ after the release of block ;

or start the timer $t_1 = \frac{\phi_1}{\omega}$ (seconds) after the release of block, etc.

This corresponds to sliding the "y-axis" bar along the various positions of the horizontal axis of the plot. (t-axis). This is what is depicted at the bottom of Pg 19.

Again, You decided ϕ ; not the block & spring given to you.

This is why we call "C" & "ϕ" each a degree of freedom.

~~① & ② on pg 18 are the two of three properties (3rd being that there's some force acting on the system (e.g. system of block + spring certainly has force involved: spring exerts force on block.)) that I said every object oscillating (simple Harmonically) must obey. Indeed, we have just proven that claim!~~

Introduction to $\mathbb{C}$ (complex) numbers:

- Before delving further into our study of oscillations and waves, we need to add a very important mathematical tool to our toolkit: Complex ($\mathbb{C}$) numbers.

  - Real numbers ($\mathbb{R}$) are #'s like $\pi, e, 1, 3, 3/2, 0, \sqrt{2}, \sqrt{3}, \ldots$
    (Basically, all the #'s you've been working with in high school.)

  But often, we naturally encounter numbers such as $\sqrt{-1}, \sqrt{-3}, \sqrt{-5}, \sqrt[4]{-7}$, etc.

  e.g. In solving $x^2+1=0$, we get $x = \pm\sqrt{-1}$.

  Numbers such as these are called "Complex" numbers. (denoted $\mathbb{C}$).

Notation: $\boxed{\text{We write } i = \sqrt{-1}.}$ ← This is called the "imaginary" number

e.g.

$$i^2 = \left(\sqrt{-1}\right)^2 = -1, \quad \sqrt{-3} = \sqrt{3}\sqrt{-1} = \sqrt{3}\,i$$

$$3\sqrt{-1} = 3i$$

$$2+\sqrt{-2} = 2+\sqrt{2}\,i, \quad \pi + b\sqrt{-1} = \pi + bi \quad (b \text{ is some } \#)$$

In general $\mathbb{C}$ number z is written as: $\boxed{z = a + bi}$, where "a" and "b" are both real numbers.

"a" is "real part" "b" is the "imaginary part".

· Some properties of $\mathbb{C}$ numbers:

Let $x = a+bi$ and $y = c+di$ be two complex #'s.

[where $a, b, c, d \in \mathbb{R}$.] (i.e; a, b, c, and d are real #'s.)

Then,

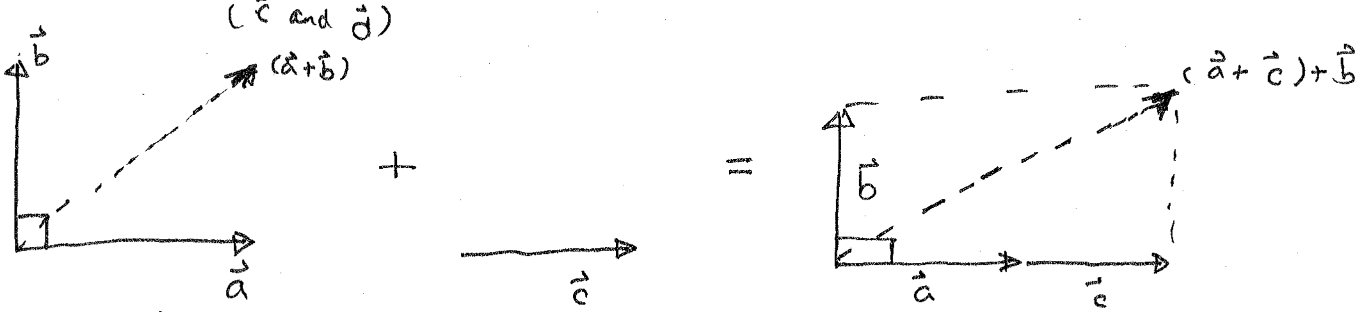
1. $(a+bi) + c = (a+c)+bi$ ← doesn't affect imaginary part "b".
2. $(a+bi) + di = a + (b+d)i$ ← doesn't affect real part "a"
3. $(a+bi) + (c+di) = (a+c) + (b+d)i$ ← Real parts (a & c) add up together. Imaginary parts (b & d) add up together. separately.

But notice that above properties of $\mathbb{C}$ numbers are very similar to the following properties of vectors in xy-plane (2 Dimensions):
[i.e. Cartesian coordinate plane]

[Reminder of rules for adding vectors in 2 dimensions]:

Let $\vec{a}$ and $\vec{b}$ be vectors: ~~are~~.
($\vec{c}$ and $\vec{d}$)

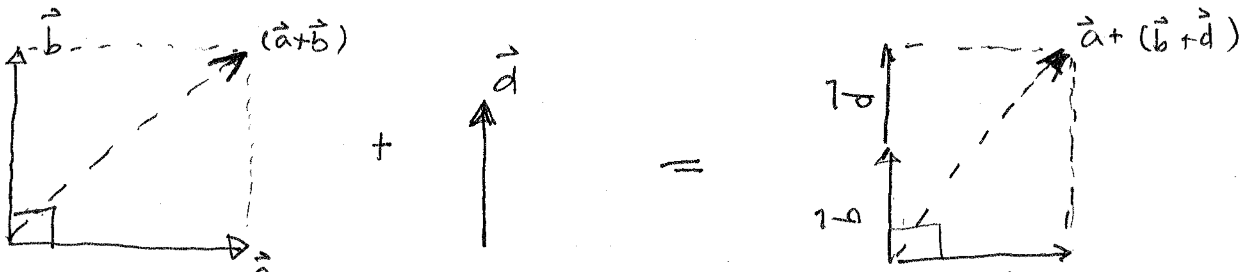
1.



$$(\vec{a}+\vec{b}) + \vec{c} = (\vec{a}+\vec{c})+\vec{b}$$

(In 1, vertical component is unaffected by the addition.)

2.

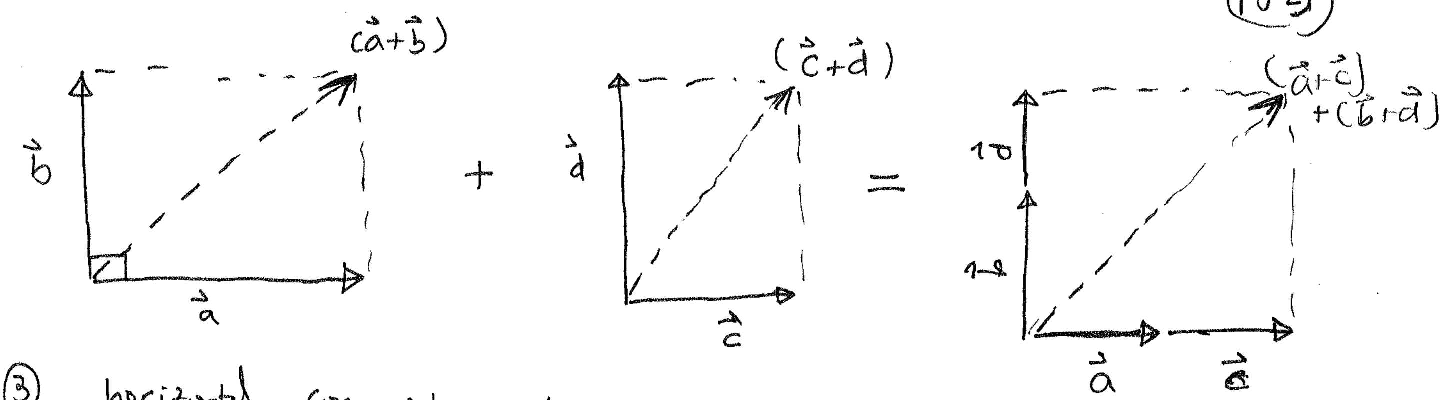


$$(\vec{a}+\vec{b}) + \vec{d} = \vec{a} + (\vec{b}+\vec{d})$$

(In 2, horizontal component is unaffected by the addition.)

over →

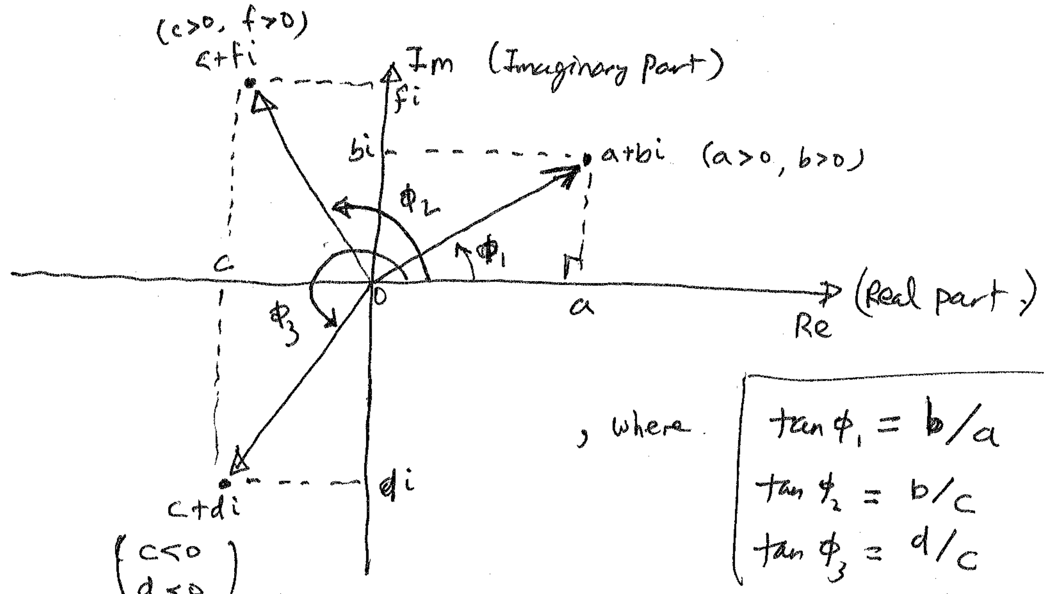
(3)



(In (3), horizontal components add together, vertical components add up together separately.)

∴ Vector addition in xy-plane (i.e. 2 dimensions) is just like adding 2 $\mathbb{C}$ numbers together.

Motivated by this, we represent $\mathbb{C}$ numbers diagramatically in a 2D plane (cartesian coordinates) as follows:



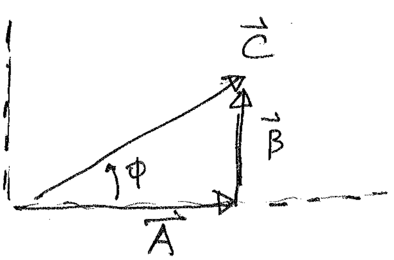
, where

$$\boxed{\begin{aligned} \tan\phi_1 &= b/a \\ \tan\phi_2 &= b/c \\ \tan\phi_3 &= d/c \end{aligned}}$$

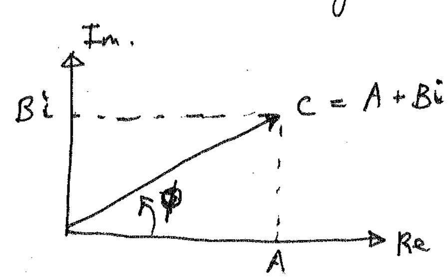
· Going back to (pg 16): we had the relationships $\begin{cases} C\cos\phi = A \\ C\sin\phi = B. \end{cases}$

These relationships are very much like the ones shown in the diagram above:

Vectors:
$|\vec{A}| = |\vec{c}|\cos\phi$
$|\vec{B}| = |\vec{c}|\sin\phi$



$\longleftrightarrow$ like



$\mathbb{C}$ #:
$A = |c|\cos\phi$
$B = |c|\sin\phi.$

- So, motivated by the common properties that vectors and $\mathbb{C}$ numbers share in common, we'll consider "C" as a $\mathbb{C}$ #.

$$C = A + Bi \quad (A, B \text{ are real \#'s}).$$

But wait, "C" is supposed to be the amplitude of oscillation, It surely cannot be $\mathbb{C}$!! Furthermore, if C is complex #, then so is X(t) since $X(t) = \underset{\mathbb{C}\ (\text{complex})}{C} \underbrace{\cos(\omega t - \phi)}_{\mathbb{R}\ (\text{real})}$, but this can't be since X(t) is a position of particle, which must be real !!!

You're absolutely right! Any physical quantity we can measure must be a real number. The point of introducing $\mathbb{C}$ numbers is that it will merely be a mathematical tool to greatly simplify calculations (Helps in making calculations shorter, less time taken to write) and the end of our calculation, we will extract only the real number part as the physically meaningful part of the answer.

- Forgetting the physics (i.e. block-spring) for now, let's say

$$C = A + Bi.$$

Then, it follows: $A = |C|\cos\phi$, $B = |C|\sin\phi$. $\left[\ |C| = \sqrt{a^2+b^2}\ .\ \right]$

magnitude of $\mathbb{C}$ number C. (Can be thought of as the length of vector $\vec{C}$. $= |\vec{C}|$.)

$$\Rightarrow \quad C = \underbrace{|C|\cos\phi}_{\text{real part}} + i\underbrace{|C|\sin\phi}_{\text{Im. part}}$$

$$\Rightarrow \quad \boxed{C = |C|\{\cos\phi + i\sin\phi\}}$$

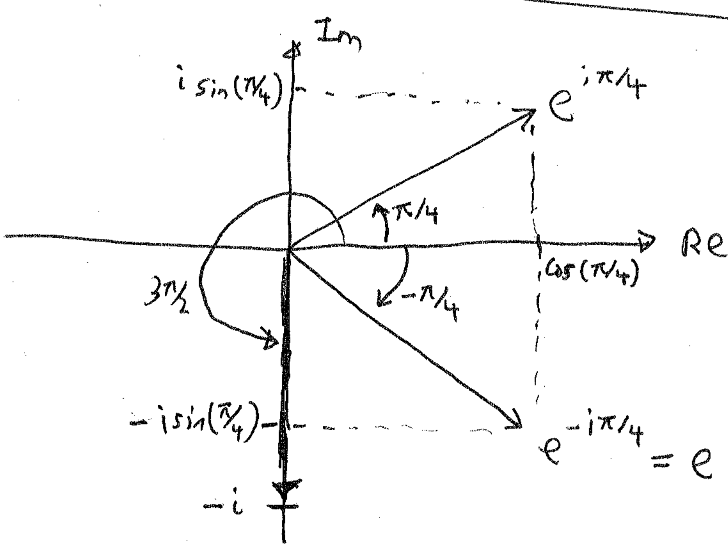
It turns out that $\boxed{\cos\phi + i\sin\phi = e^{i\phi}}$ ← aka. "De Moivre's theorem"

so: $C = |C| \{\cos\phi + i\sin\phi\}$
$= |C| e^{i\phi}$

[<u>Note</u>: One way to <u>prove</u> that $e^{i\phi} = \cos\phi + i\sin\phi$ is through Taylor series of $e^{i\phi}$, $\sin\phi$, and $\cos\phi$. We will not prove this identity here. Unfortunately, we'll just take it for granted that indeed

$$\boxed{e^{i\phi} = \cos\phi + i\sin\phi}$$
]

• Notice that $\boxed{|e^{i\phi}| = \sqrt{\cos^2\phi + \sin^2\phi} = 1}$



• Motivated by the fact that $x(t) = C\cos(\omega t - \phi)$
$= A\cos(\omega t) + B\sin(\omega t)$ (see Pg 16),

let's consider the $\mathbb{C}^{\#}$ analogue of this: by defining:

$$\boxed{z(t) \equiv C e^{i(\omega t - \phi)}}$$

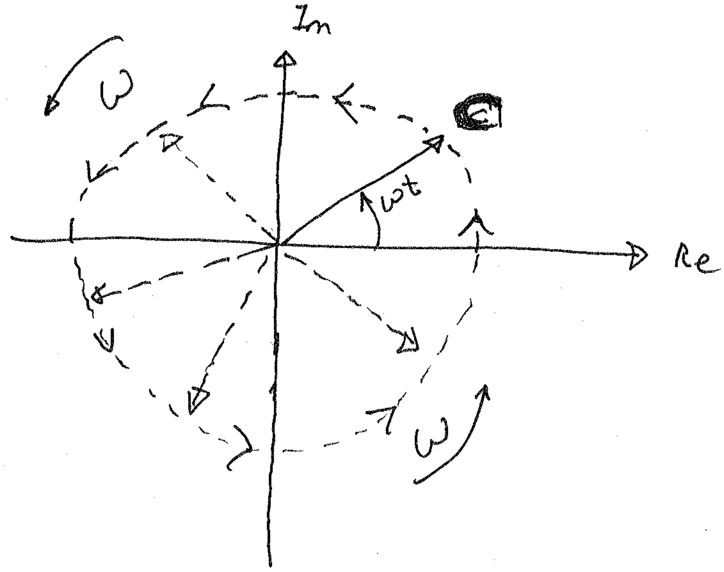
"define as"

Notice that: since $C = |C| e^{i\phi}$ (see top of this page)

we have: $Z(t) = C e^{i(\omega t - \phi)}$
$= |C| e^{i\phi} e^{i\omega t} e^{-i\phi}$ ← ∵ $\left(e^{y_1 + y_2} = e^{y_1} e^{y_2}\right)$
$= |C| \underbrace{e^{i\phi - i\phi}}_{e^0 = 1} e^{i\omega t} = |C| e^{i\omega t}$
$= |C|\cos(\omega t) + i|C|\sin(\omega t)$.

and notice that $Z(t) = |C| \cos(\omega t) + i|C|\sin(\omega t)$ is describing a "vector" (arrow) of length $|C|$, that is rotating counterclockwise with angular frequency (velocity) ω.

i.e.



And indeed, $z(t)$ as written above looks like $x(t)$ but with an imaginary part.

Now, similarly, ~~[illegible]~~ motivated by the SHM EOM: $\frac{d^2x}{dt^2} + \omega^2 x = 0$, let's

Consider $\underbrace{\frac{d^2 Z}{dt^2} + \omega^2 Z}$ and see if this is zero.

First, simplify notation a bit: $Z(t) = C e^{i(\omega t - \phi)}$

$$= \underbrace{C e^{-i\phi}}_{\substack{\parallel \\ C_0}} e^{i\omega t} = C_0 e^{i\omega t}$$

← define (so we don't have to keep writing $e^{-i\phi}$.)

Then:

$$\frac{d^2 Z}{dt^2} + \omega^2 Z = -\omega^2 \overbrace{C_0 e^{i\omega t}}^{\parallel\, Z(t)} + \omega^2 Z$$

plug in

$$= \underbrace{-\omega^2 Z} + \underbrace{\omega^2 Z}$$

cancel each other out

$$= 0.$$

$$\frac{dZ}{dt} = \frac{d}{dt}(C_0 e^{i\omega t}) = C_0 i\omega e^{i\omega t} = i\omega Z(t)$$

$$\frac{d^2 Z}{dt^2} = i\omega \frac{dZ}{dt} = i\omega C_0 i\omega e^{i\omega t} = -\omega^2 C_0 e^{i\omega t}$$

← $\because i^2 = \sqrt{-1}^2 = -1$.

$\therefore$ Indeed, $\frac{d^2 Z}{dt^2} + \omega^2 Z = 0$, just like our SHM EOM.

- This is why there's such an intimate connection between a uniform circular motion ($z(t)$ in $\mathbb{C}$-plane) and SHM ($x(t)$ on real line).

* From now on, we will work with the $\mathbb{C}$ analogue $z(t)$ instead of $x(t)$ to do calculations ($\because$ less time to write $e^{i\omega t}$ than $\sin(\omega t)$ and also, our calculations will become much simpler.)

- And when we're done with calculation, at the end, we will just extract the real part of $z(t)$ as the physically meaningful component. $x(t)$.
  (since position has to be $\mathbb{R}$ not $\mathbb{C}$.)

i.e. procedure from now on

step 1.) Derive EOM of the physical system of interest.
(e.g. $\frac{d^2x}{dt^2} + \omega^2 x = 0$) $x(t) \in \mathbb{R}$

2.) We will then write down the $\mathbb{C}$ - equivalent version of EOM:

$$\left(\text{e.g. } \frac{d^2 z}{dt^2} + \omega^2 z = 0 \right) \quad z(t) \in \mathbb{C}$$

3) We will solve for $z(t) \in \mathbb{C}$. (e.g. $z(t) = C_0 e^{i\omega t}$)

4) To be physically meaningful, need real # description of the world. so, only take out the real part of $z(t)$.
(quote)
This would be $x(t)$. (i.e. $\boxed{x(t) = \mathrm{Re}\,(z(t))}$.)