Forced Oscillations:

June 27, 2008
[Fri]
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(pg 39)

- So far we have considered simple harmonic motion with
  1. restoring force. (No friction) (simplest case)
  2. restoring force + friction (damping)

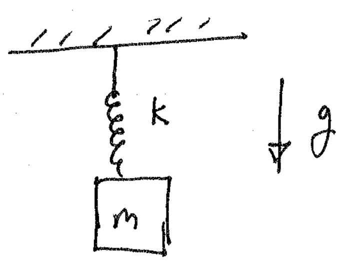
- Today, we study oscillations with restoring force (proportional to displacement from equilibrium)
  + damping force (proportional to velocity of particle)
  + other types of force (Namely, external force ~~that~~, a function of time t.)

  = Forced oscillator.

- Let's first ignore damping force, and only consider a very simple external force: a constant force acting on a particle:

Case 1: Constant external force: We've actually encountered a situation before, in which there was a force other than the spring force, acting on a block (No friction); and in which that force was constant.

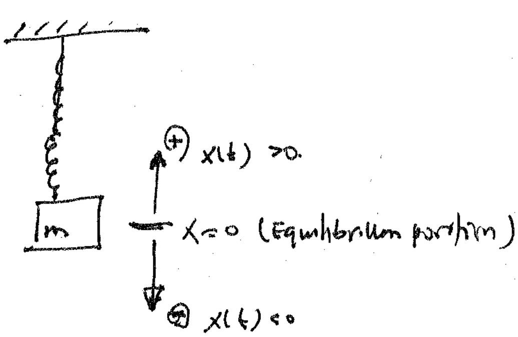
I'm talking about:



spring force (restoring force) + gravity mg $= \vec{F}_{net}$.

- But we already know how the block moves: just a simple Harmonic motion (SHM).
- We've been a little bit coy all this time; we always chose to describe motion of block using $x(t) \equiv$ displacement from the equilibrium position

i.e.

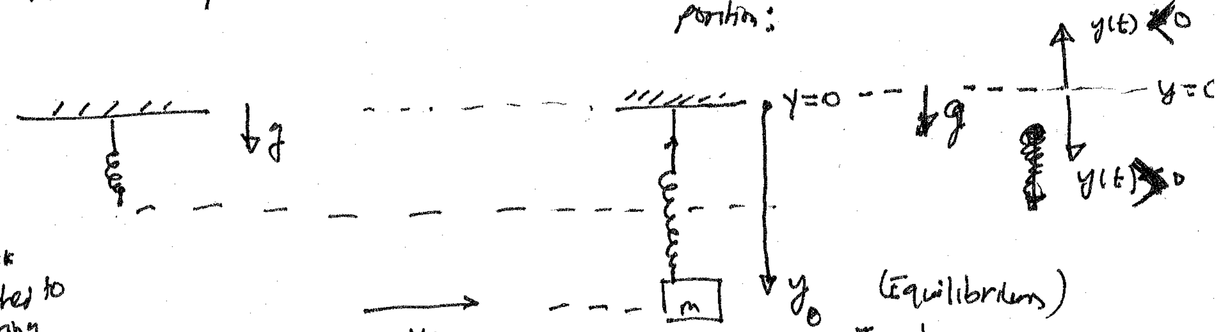


← In this picture, we didn't have to think about the effect of gravity on the block. Why?

over →

<u>Answer</u>: To answer this question, let's work with $y(t) \equiv$ position of block measured relative to the ceiling instead of $x(t) \equiv$ displacement of block from equilibrium position:

<u>Imagine</u> initially,

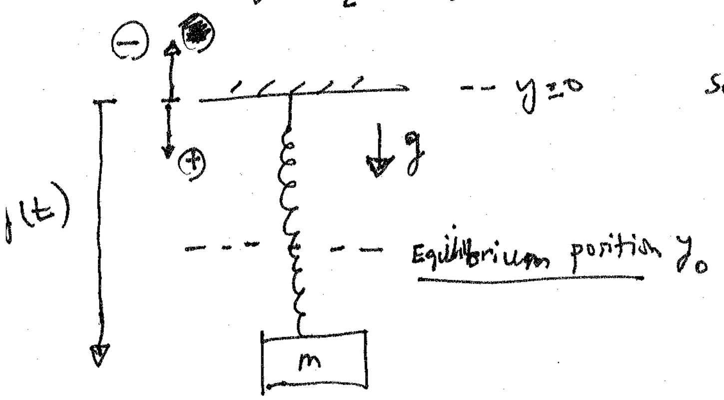


No block attached to spring. (Spring is at its rest length).

then, attach a block to spring

The spring stretches and equilibrium is reached at position $y = y_0$. $(y_0 > 0)$.

Next, grab the block from it's equilibrium, pull it down by some distance "A" ← (amplitude) then release the block. The subsequent motion of block is described by the following EOM:



So:

$$m\ddot{y} = \underbrace{-ky}_{\text{restoring force}} + mg$$

downwards is measured to be ⊕ here since we set $y > 0$ below the ceiling.

mg: Constant gravity on block

$$\Downarrow$$

$$m\ddot{y} = -k\left(y - \frac{mg}{k}\right)$$

$$\Rightarrow \ddot{y} + \frac{k}{m}\left(y - \frac{mg}{k}\right) = 0$$

But $\frac{mg}{k} = y_0$ ← equilibrium position.

So $y(t) - \frac{mg}{k}$ represents <u>displacement</u> from equilibrium.

Let $x(t) \equiv y(t) - \frac{mg}{k}$.

then we now have: $\ddot{y} + \frac{k}{m}x = 0$.

But $\ddot{x} = \ddot{y}$ (since mg/k is just a constant)

so: we get: $\boxed{\ddot{x} + \frac{k}{m}x = 0}$ ← Exactly what we've been implicitly using all this time.

We've just studied a particular example ~~ex~~ involving a constant force (gravity mg in the example) but the EOM we would get for any other system involving : restoring force (proportional to displacement) + constant force (and no damping) would "look" the same as the EOM of previous example :

we'd get : $$\boxed{\ddot{y} + \omega_0^2\left(y - \frac{F_0}{m\omega_0^2}\right) = 0} \quad \ldots \boxed{\text{Eq'n (1)}}$$

(where $F_0 \equiv$ constant force exerted on a particle)

(e.g. for the previous example, we had $F_0 = mg$.

so : $$\frac{F_0}{m\omega_0^2} = \frac{mg}{m\,k/m} = mg/k \equiv y_0 \leftarrow \text{equilibrium position.}$$

That is, above eq'n (1) can be rewritten as :

$$\ddot{y} + \omega_0^2(y - y_0) = 0 \qquad \text{where } y_0 \equiv \underline{\text{Equilibrium position}}$$

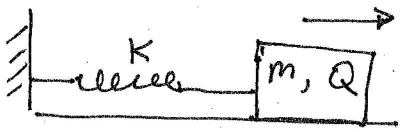
So, the only effect of a <u>constant force</u> is in determining the equilibrium position ; but beyond that, we don't need to think about the constant force any further : we can just think about <u>displacement</u> from the equilibrium position since above equation can be rewritten only in terms of displacement $x(t) \equiv y(t) - y_0$ as : (since $\ddot{x} = \ddot{y}$).

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \quad \leftarrow \text{usual } \underline{\text{SHM eq'n}}$$

Case 2: More interesting t-dependent force: $\boxed{F_{external}(t) = F_0 \cos(\omega t)}$ (sinusoidally varying external force)

First, consider the case w/o friction.

Ex: Consider: (see footnote*)



$E(t) = E_0 \cos(\omega t)$

No friction.

mass $= m$
Electric charge $= Q$.
$E(t) = E_0 \cos(\omega t)$

↑ Electric field, varying w/ time (but not with respect to position.)

Then, the force felt by the block is:

1. restoring force: $-kx$
2. External force: $F(t) = QE(t)$

$$= QE_0 \cos(\omega t)$$
$$= F_0 \cos(\omega t) \text{ , where } F_0 \equiv QE_0.$$

↑ Amplitude of force (i.e. maximal external force that would act on block)

$x(t) \equiv$ Displacement from equilibrium, as usual.

So, EOM is:

$$m\ddot{x} = -kx + F_{ext.}(t)$$
$$= -kx + F_0 \cos(\omega t)$$
$$\Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t) \text{ ; let } \omega_0^2 = k/m$$

$\omega_0 \equiv$ "Natural angular frequency"

↑ angular frequency with which the block will oscillate in the absence of external force.

$$\Rightarrow \boxed{\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)}$$

$\boxed{\text{Eq'n (2)}}$

↑ EOM for our forced oscillator

(Notice that ω can be different from ω_0).

We want to solve this EOM:

→ over

Footnote: Technically, our example here is incorrect. When a charge accelerates (as our block is doing all the time), the charge would radiate away energy. ~~This is~~ This would be a "damping" term that we neglected above.

Let's consider a guess : $X_p(t) \equiv A\cos(\omega t)$
(use "guess & check" method of solving EOM)

Then plugging into EOM:

$$\ddot{X}_p + \omega_0^2 X_p = -\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t)$$
$$= \underline{\left[\omega_0^2 - \omega^2\right] A\cos(\omega t)}$$

We want this to equal:

$$(\omega_0^2 - \omega^2) A \cancel{\cos(\omega t)} = \frac{F_0}{m}\cancel{\cos(\omega t)}$$

$$\Rightarrow \quad A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

(Assuming that $\omega_0 \neq \omega$). Notice that if $\omega_0 = \omega$, this can't be a sol'n since $\frac{1}{\omega_0^2 - \omega^2} \to \infty$ as $\omega \to \omega_0$.

so: $\boxed{X_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)}$ is a sol'n to forced ~~SHM~~ SHM EOM.

Strange: This solution has no free parameters. But since ~~our~~ EOM is a 2nd order differential eq'n, we know that the ~~solution~~ most general solution has to have 2 free parameters.
(Amplitude "C" & phase shift "ϕ").

What we found above: $X_p(t)$: is a "particular solution"
(thus the subscript "p")

- It's a "particular" solution since the amplitude $\left(\frac{F_0}{m(\omega_0^2 - \omega^2)}\right)$ and phase constant ($\phi = 0$) have been already picked out for you. (i.e. ~~It's a particular solution~~
It's describing the particular situation in which $\frac{F_0}{m(\omega_0^2 - \omega^2)}$ is amplitude, and phase shift is zero.)

Now, find the general sol'n:

- Going back to pg 42,

our EOM is: $\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$.

But notice that: $\ddot{x} + \omega_0^2 x$ is a linear differential eq'n.

- That is, if I plug in $x = x_1 + x_2$,

I get: $\underbrace{(\ddot{x}_1 + \omega_0^2 x_1)} + \underbrace{(\ddot{x}_2 + \omega_0^2 x_2)}$

Addition of 2 versions of our original differential eq'n. (One for x_1, the other for x_2).

- Keeping this in mind, notice that if we let $x_{SHM}(t) \equiv C\cos(\omega_0 t - \phi)$

and $x_2(t) \equiv x_p(t)$, then:

let $x(t) = x_{SHM} + x_p$,

we'd get: $\ddot{x} + \omega_0^2 x = \underbrace{\cancel{(\ddot{x}_{SHM} + \omega_0^2 x_{SHM})}}_{0} + \underbrace{(\ddot{x}_p + \omega_0^2 x_p)}_{= \frac{F_0}{m}\cos(\omega t)}$

$\because x_1(t)$ is a sol'n to SHM EOM.

(found before)

$= \frac{F_0}{m}\cos(\omega t)$

Hence, using the superposition (aka linearity) principle of EOM,

we have just found that

$$\boxed{\begin{aligned} x(t) &= x_{SHM}(t) + x_p(t) \\ &= C\cos(\omega_0 t - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \end{aligned}}$$

is a sol'n to EOM.

In fact, this must be the general sol'n to EOM since it has

2 free parameters: C ≡ Amplitude you decide on.

ϕ ≡ phase constant you decide on.

• Let's look at the behavior of $X(t) \equiv$ position of our charged block (Q, M).

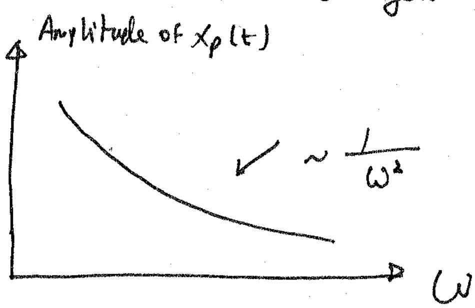
We look at the following 2 "limiting" cases:

<u>Limiting case 1</u>: <u>Large angular frequency ω of force</u>: $\underline{\omega \gg \omega_0}$.

Then $$X_p(t) = \frac{F_0 \cos(\omega t)}{m(\omega_0^2 - \omega^2)} \underset{(\omega \gg \omega_0)}{\approx} -\frac{F_0 \cos(\omega t)}{m\omega^2}$$

so, as $\omega \to +\infty$, $X_p(t)$ decays like $\sim \frac{1}{\omega^2}$

(i.e. Amplitude of $X_p(t)$ is $\frac{-F_0}{m\omega^2}$, which decays as $\frac{1}{\omega^2}$ as ω gets large.)



• <u>Why?</u> <u>We want a physical explanation</u>:

<u>Ans</u>: Large ω corresponds to fast oscillation of external force $F_0 \cos(\omega t)$ between its ~~amplitudes~~ 2 extreme values: $+F_0$ and $-F_0$.

(recall that period of oscillation is $\left(T = \frac{2\pi}{\omega}\right)$. so large ω, $\Rightarrow$ small T.

When the oscillation of the force becomes so fast ($\omega \gg \omega_0$; $\omega \to \infty$), then the block feels an <u>average force</u>; which in our case is <u>zero</u> (since $F(t) = F_0 \cos(\omega t)$)

Hence, the effect of external force becomes more and more irrelevant as ω gets larger & larger.

(i.e. $X_p \to 0$.)

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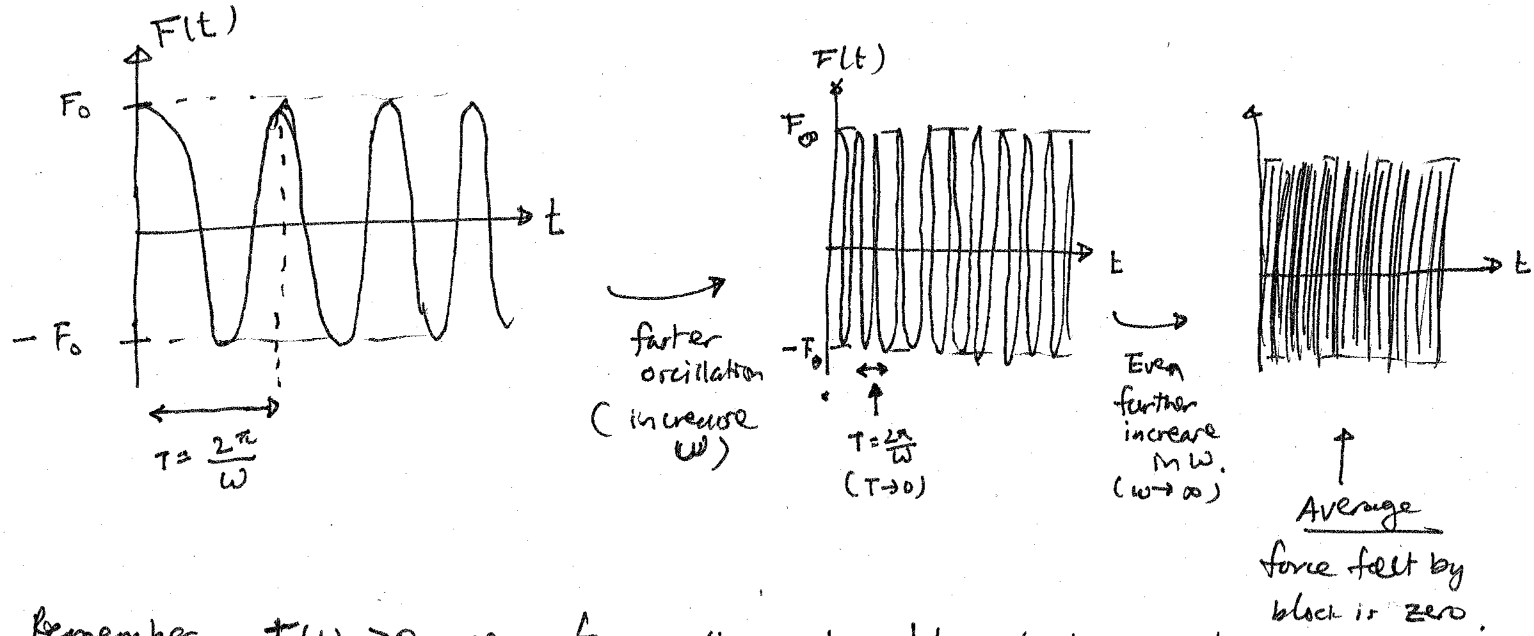
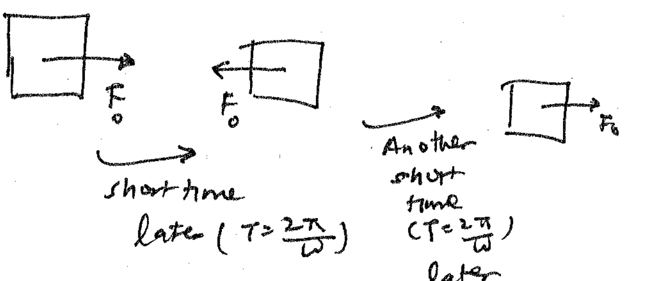
· Graphically, we can ~~[illegible]~~ see this:



Remember, $F(t) > 0$ means force acting on the block to the right.
and $F(t) < 0$ means force acting on the block to the left. (given our sign convention for x on pg 42)

So: for large ω:



and so on. Due to this flipping back & forth (very fast flips), the block feels avg. force $= 0$,

So, in this case, $x(t) \xrightarrow[\text{Approaches}]{} x_{hom}(t)$ (since $x_p(t) \to 0$).

Limiting case 2: Slow oscillation of force: $0 < \omega \ll \omega_0$.

Here, $x_p(t) = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \underset{(\omega_0 \gg \omega)}{\approx} \dfrac{F_0 \cos(\omega t)}{m\omega_0^2}$.

So:

$$x(t) \approx \underbrace{C \cos(\omega_0 t - \phi)}_{\text{oscillates w/ period } \frac{2\pi}{\omega_0}} + \frac{F_0}{m\omega_0^2} \underbrace{\cos(\omega t)}_{\text{oscillates w/ period } \frac{2\pi}{\omega}}.$$

much Faster oscillation than this.

Subcase 2.1 : If $\omega_0 \gg \omega$, and $F_0 \gg m\omega^2$ (Note $m\omega^2 = k$) ← spring constant

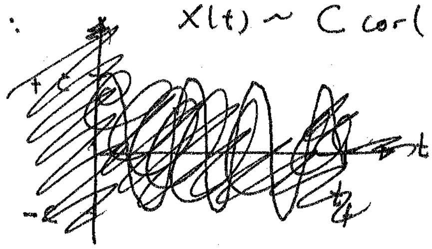
~~so~~, large natural angular frequency ω_0 and very strong force F_0):

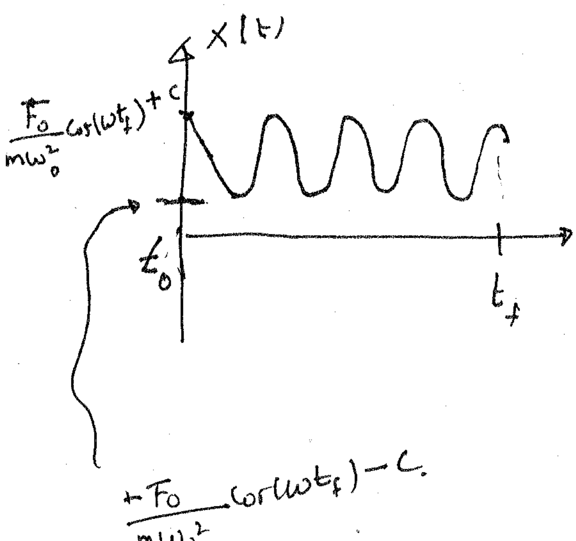
then ~~we~~ have:
$$x(t) \approx \underbrace{C\cos(\omega_0 t - \phi)}_{\text{fast oscillation.}} + \underbrace{\left(\frac{F_0}{m\omega_0^2}\right)}_{\text{large amplitude}} \underbrace{\cos(\omega t)}_{\text{slow oscillation.}}$$

then, if you pay attention to a time interval much smaller than $T' \equiv \frac{2\pi}{\omega}$ (say time interval on the order of $T_0 \equiv \frac{2\pi}{\omega_0}$ (recall, here: $T' \gg T_0$))

then $\frac{F_0}{m\omega_0^2}\cos(\omega t)$ looks like a constant on this short time interval.

And we'd see approximately: $x(t) \sim C\cos(\omega_0 t - \phi) + \left(\frac{F_0}{m\omega_0^2}\right)\cos(\omega t_f)$

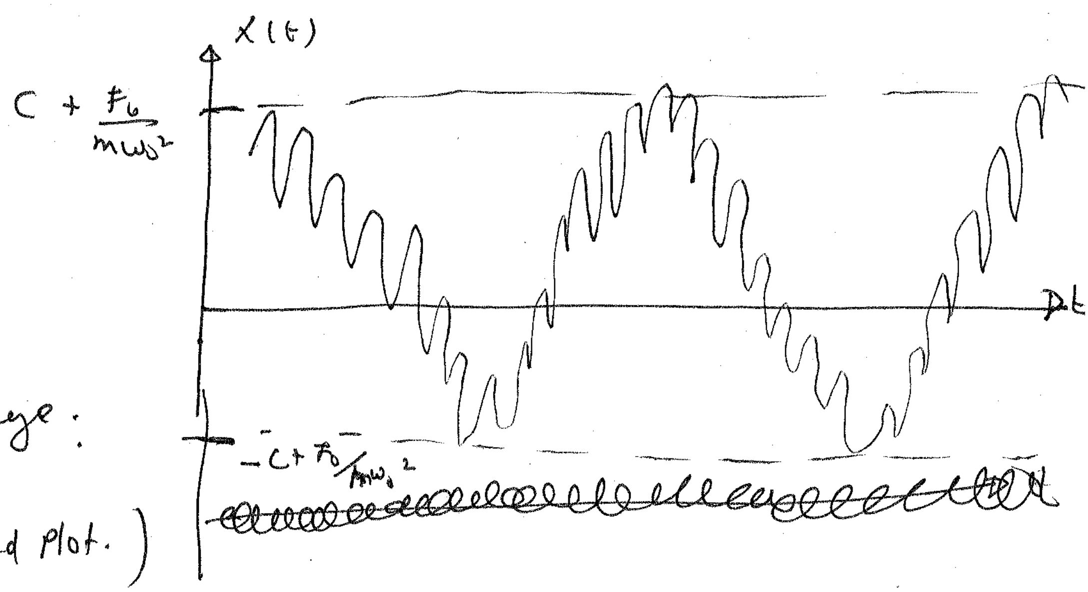




$t_f \ll T'$

t_f on the order of T_0.
(say 2~3 times T_0)

But on a time scale comparable to the large T', we'd see:

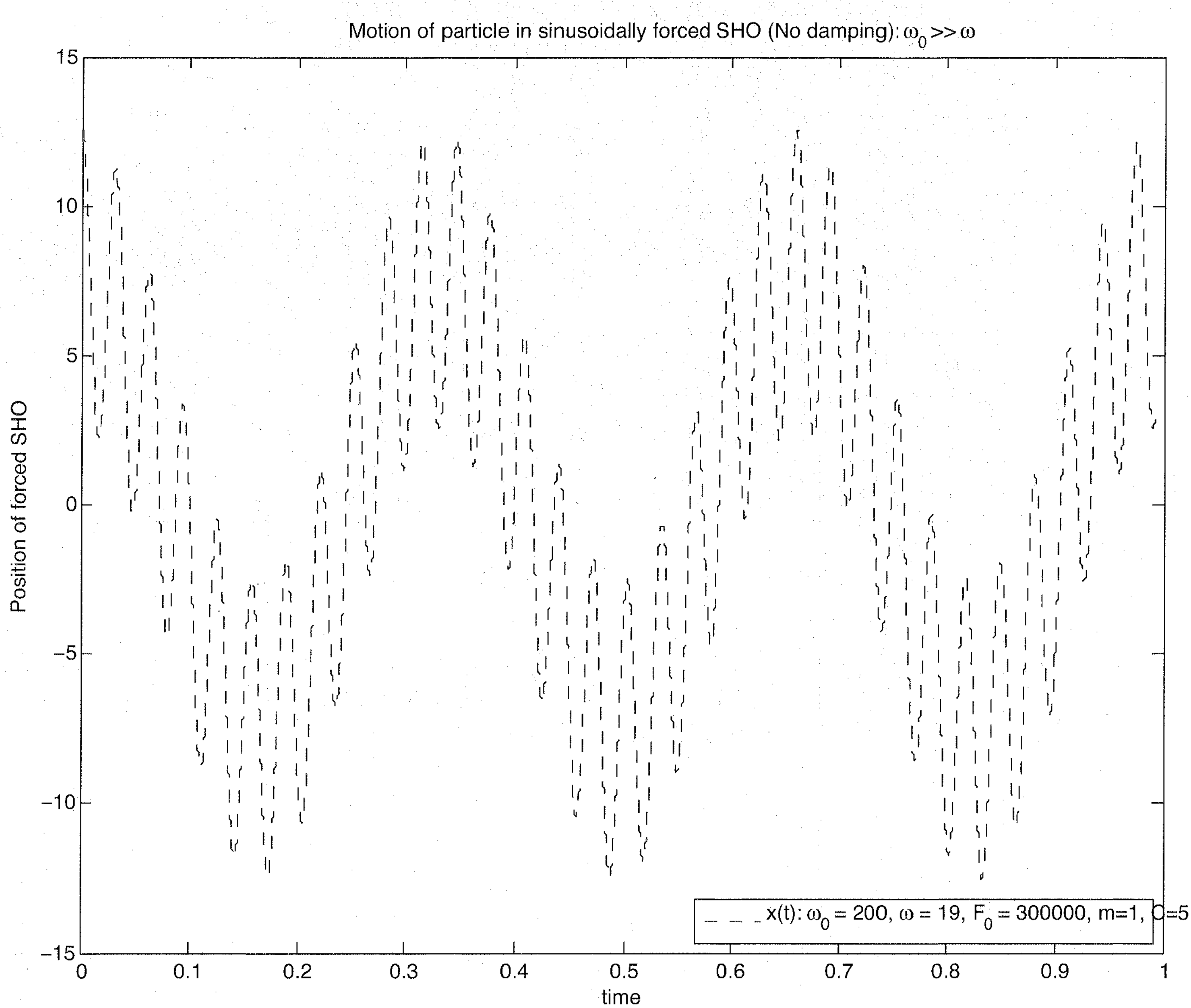


(see next page: for a neat MATLAB generated plot.)

Limiting case:

Large natural frequency
and large force amplitude

$$\left\{ \begin{array}{l} \omega_0 \gg \omega \\ F_0 \gg m\omega_0^2 \end{array} \right\}.$$



Subcase 2.2 : $|F_0| << m\omega_0^2$, and $\omega_0 >> \omega$:

(small force)

Then $x(t) \sim C\cos(\omega_0 t - \phi)$ (can ignore the effect of force : $x_p(t)$).

* Basically, the physical idea in the case where $\omega_0 >> \omega$ is that since the period associated w/ natural angular frequency ω_0 is $T_0 \equiv \frac{2\pi}{\omega_0}$ is much smaller than that associated w/ the force : $T = \frac{2\pi}{\omega}$, then on a time scale much shorter than T, we can treat the external force as constant.

Resonance : What happens as ω_0 approaches ω?

As ω approaches ω_0, $x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \to \infty$.

($\because \frac{1}{\omega_0^2 - \omega^2} \to \infty$)

This is called resonance.

Before delving into resonance any further, let's first investigate the most general scenario: A physical system w/ restoring force
+ damping
+ external force that sinusoidally varies.

EOM :
$$\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)}$$
← EOM

How to solve this EOM?

Ans : Use linearity principle again. over

(a.k.a. : "superposition principle.")

· But first, find the particular soln to EOM : $X_p(t)$:

To do this, let's solve the $\mathbb{C}$- analogue of EOM :

$\mathbb{R}$: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$ But, $\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$

$$= \frac{F_0}{m}\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

(recall : $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

so, $e^{-i\omega t} = \cos(-\omega t) + i\sin(-\omega t)$
$= \cos(\omega t) - i\sin(\omega t)$)

To get $\mathbb{C}$-# analogue of EOM, just change : $x \to z$:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \frac{F_0}{2m}\left[e^{i\omega t} + e^{-i\omega t}\right] \leftarrow \text{$\mathbb{C}$ - equivalent EOM,}$$

, To solve, "guess and check" method is used.

· But first, notice that we can think of above eq'n as 2 separate linear differential equations added together :

$$\ddot{z}_1 + 2\gamma\dot{z}_1 + \omega_0^2 z_1 = \frac{F_0}{2m}e^{i\omega t} \quad \cdots (1)$$

and

$$\ddot{z}_2 + 2\gamma\dot{z}_2 + \omega_0^2 z_2 = \frac{F_0}{2m}e^{-i\omega t} \quad \cdots (2)$$

Adding eq'ns (1) & (2) we get :

original EOM $\rightarrow$ $$\underbrace{(\ddot{z}_1 + \ddot{z}_2)}_{\ddot{z}_p} + 2\gamma\underbrace{(\dot{z}_1 + \dot{z}_2)}_{\dot{z}_p} + \omega_0^2\underbrace{(z_1 + z_2)}_{z_p} = \frac{F_0}{2m}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

← "linearity" (aka. "superposition") principle.

* We look for particular soln $z_p(t) = z_1 + z_2$ first.

So, what are $z_1(t)$ and $z_2(t)$?

· Guess & check method of solving:

Guess: $z_1(t) = Ae^{i\omega t}$. (Motivated by how we solved no damped ($\gamma=0$) forced SHM eq'n on pg 43)

Then: Check:

$$\ddot{z}_1 + 2\gamma\dot{z}_1 + z_1\omega_0^2 = -\omega^2 z_1 + 2\gamma i\omega z_1 + z_1\omega_0^2$$

$$= (-\omega^2 + 2\gamma i\omega + \omega_0^2)\, z_1$$

we want this to equal to:

$$(-\omega^2 + 2\gamma i\omega + \omega_0^2)\cancel{A}\cancel{e^{i\omega t}} = \frac{F_0}{2m}\cancel{e^{i\omega t}}$$

$\Rightarrow$ Need:
$$\boxed{A = \frac{F_0}{2m[\omega_0^2 - \omega^2 + 2i\gamma\omega]}}$$

(As w/ the $\gamma=0$, we found the necessary amplitude "A" for the particular sol'n $z_1(t)$.)

Similarly:
$z_2(t) \equiv Be^{-i\omega t}$ and:

$$\boxed{B = \frac{F_0}{2m[\omega_0^2 - \omega^2 - 2i\gamma\omega]}}$$ ← (Just by changing: $\omega \to (-\omega)$.)

so: $z_p(t) = z_1(t) + z_2(t)$

$$= \frac{F_0}{2m}\left\{\frac{e^{i\omega t}}{[\underbrace{(\omega_0^2-\omega^2)}_{\equiv\,\Omega^2} + 2i\gamma\omega]} + \frac{e^{-i\omega t}}{[\underbrace{(\omega_0^2-\omega^2)}_{\equiv\,\Omega^2} - 2i\gamma\omega]}\right\}$$

← define as

over →

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so: $$Z_p(t) = \frac{F_0}{2m}\left\{ \frac{e^{i\omega t}(\Omega^2 - 2i\gamma\omega)}{[\underbrace{(\omega_0^2-\omega^2)}_{\equiv\,\Omega^2} + 2i\gamma\omega](\Omega^2 - 2i\gamma\omega)} + \frac{e^{-i\omega t}(\Omega^2 + 2i\gamma\omega)}{[\underbrace{(\omega_0^2-\omega^2)}_{\equiv\,\Omega^2} - 2i\gamma\omega](\Omega^2 + 2i\gamma\omega)} \right\}$$

Multiply top & bottom of each term by complex conjugate

$$= \frac{F_0}{2m}\left\{ \frac{e^{i\omega t}(\Omega^2 - 2i\gamma\omega) + e^{-i\omega t}(\Omega^2 + 2i\gamma\omega)}{\Omega^4 + 4\gamma^2\omega^2} \right\}$$

$$= \frac{F_0}{2m}\left\{ \frac{\Omega^2 \overbrace{[e^{i\omega t} + e^{-i\omega t}]}^{=\,2\cos(\omega t)} - 2i\gamma\omega \overbrace{[e^{i\omega t} - e^{-i\omega t}]}^{=\,2i\sin(\omega t)}}{\Omega^4 + 4\gamma^2\omega^2} \right\}$$

$$= \frac{F_0}{\cancel{2}m}\left\{ \frac{\cancel{2}\Omega^2\cos(\omega t) + \cancel{4}^{\,2}\gamma\omega\sin(\omega t)}{\Omega^4 + 4\gamma^2\omega^2} \right\}$$

$$\Rightarrow \boxed{Z_p(t) = \frac{F_0\,[\Omega^2\cos(\omega t) + 2\gamma\omega\sin(\omega t)]}{m\,(\Omega^4 + 4\gamma^2\omega^2)}}$$ ← $\mathbb{R}$ actually (No imaginary terms)

↳ This has <u>no free parameters</u> ("Particular" sol'n).

The missing part of sol'n (w/ 2 free parameters) comes from the "damped SHM" part.

That is: $$\boxed{\ddot{Z} + 2\gamma\dot{Z} + \omega_0^2 Z = \frac{F_0}{2m}[e^{i\omega t} + e^{-i\omega t}]}$$ ← $\mathbb{C}$-# analogue of EOM.

• is sum of 2 eqns:

$$\ddot{Z}_{\text{damped SHM}} + 2\gamma\dot{Z}_{\text{damped SHM}} + \omega_0^2 Z_{\text{damped SHM}} = 0 \quad \text{--- (1)}$$

and, $$\ddot{Z}_p + 2\gamma\dot{Z}_p + \omega_0^2 Z_p = \frac{F_0}{2m}[e^{i\omega t} + e^{-i\omega t}] \quad \text{--- (2)}$$

So adding eqn (1) & (2); we get

$$\underbrace{(\ddot{Z}_{\text{damped}} + \ddot{Z}_p)}_{\to\,\ddot{Z}} + 2\gamma\underbrace{(\dot{Z}_{\text{damped}} + \dot{Z}_p)}_{\to\,\dot{Z}} + \omega_0^2\underbrace{(Z_{\text{damped}} + Z_p)}_{\to\,Z} = \frac{F_0}{2m}[e^{i\omega t} + e^{-i\omega t}]$$

so, letting $Z(t) = Z_{\substack{\text{damped} \\ \text{SHM}}}(t) + Z_p(t)$ solves EOM and is in fact the general sol'n since it has 2 free parameters:

(coming from $Z_{\substack{\text{damped} \\ \text{SHM}}}$)

where,

$$Z_{\substack{\text{damped} \\ \text{SHM}}}(t) = e^{-\gamma t}\left[C_1 e^{i\tilde{\omega} t} + C_2 e^{-i\tilde{\omega} t}\right] \; ; \quad \text{where } \tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$$

$C_1, C_2 \equiv$ 2 free parameters

$\therefore$ The general sol'n to EOM is ($\mathbb{C}$-analogue):

$$Z(t) = e^{-\gamma t}\left[C_1 e^{i\tilde{\omega} t} + C_2 e^{-i\tilde{\omega} t}\right] + \frac{F_0}{m}\frac{\left[\Omega^2 \cos(\omega t) + 2\gamma\omega\sin(\omega t)\right]}{(\Omega^4 + 4\gamma^2\omega^2)}$$

* Now, to get the $\mathbb{R}$-analogue sol'n (the actual, physically meaningful part), we need to first know what regime of damping we're in.

For example, if we're in underdamped case $\therefore$ $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2} > 0 \in \mathbb{R}$.

$(\omega_0 > \gamma)$

then notice that:

$$C_1 e^{i\tilde{\omega} t} + C_2 e^{-i\tilde{\omega} t} = (C_1 + C_2)\cos(\tilde{\omega} t) + i(C_1 - C_2)\sin(\tilde{\omega} t)$$

So, $\operatorname{Re}(Z(t)) = (C_1 + C_2)\cos(\tilde{\omega} t)$

↑ we need to stick in "ϕ"

so it's actually $\underset{\substack{\parallel \\ C_1 + C_2}}{C}\cos(\tilde{\omega} t - \phi)$. $= x(t) = \operatorname{Re}(Z(t))$.

Thus,

$$\boxed{x(t) = C e^{-\gamma t}\cos(\tilde{\omega} t - \phi) + \frac{F_0}{m}\frac{\left[\Omega^2 \cos(\omega t) + 2\gamma\omega\sin(\omega t)\right]}{\left[\Omega^4 + 4\gamma^2\omega^2\right]}}$$

↑ underdamped case.

* I'll ask you to investigate how $x(t)$ found on previous (pg) behaves on Problem set 2. There, you'll learn about resonance.

$\boxed{E}$