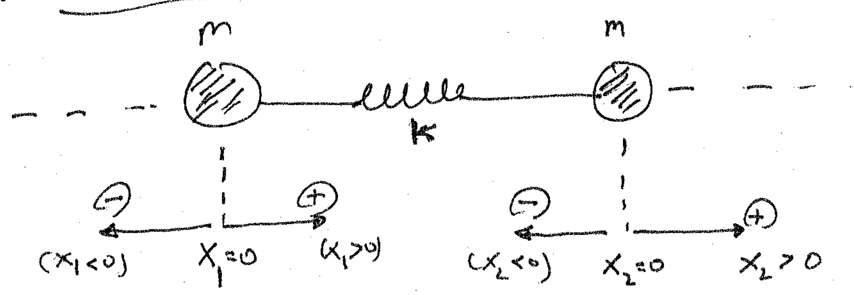
# 2. Oscillations of lots of particles together

- 2nd part of our course (see course syllabus).

- Simplest case: Consider oscillation of 2 bodies. (say 2 particles)
  Coupled to each other (i.e. interact w/ one another by exerting force on each other.)

Ex: 2 particles attached to each other through spring: constrained to move in 1D (x-axis):



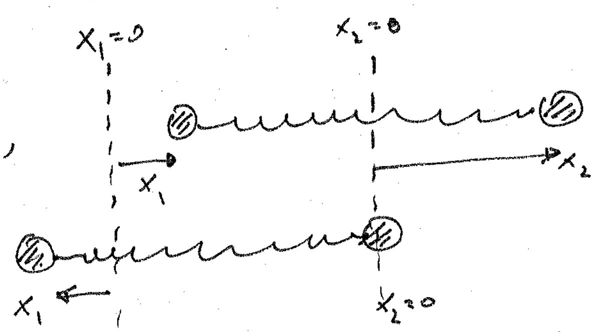

$x_1=0$ , $x_2=0$

equilibrium positions of particles 1 & 2, respectively.

- Notice that if $(x_2-x_1)>0$ : spring stretched.
- " " if $(x_2-x_1)<0$ : spring compressed
- " " if $x_2-x_1=0$ : spring is at its rest length.

e.g:

In both cases, $x_2-x_1>0$.




Now, we want to write down the EOM of this system:

- We'll write EOM for particle 1 :
- And another for particle 2 .

$$\boxed{\begin{aligned} m\ddot{x}_1 &= k(x_2-x_1). \quad \ldots (1) \\ m\ddot{x}_2 &= -k(x_2-x_1). \quad \ldots (2) \end{aligned}}$$

2 EOMs (1) & (2).

How do we solve this?

over

• Notice that: ~~addi~~

1) Adding up eqn ① and ② we get:

$$m(\ddot{x}_1 + \ddot{x}_2) = k(x_2 - x_1) - k(x_2 - x_1)$$
$$= 0.$$

$$\Rightarrow \quad \ddot{x}_1 + \ddot{x}_2 = 0$$

Letting $X \equiv x_1 + x_2$; above can be written as $\ddot{X} = 0$

The most general soln of this is: $\boxed{X(t) = At + B.}$

(check by seeing what $\ddot{X}$ is.)

Note

• To solve $\ddot{X} = 0$ systematically:

("Method 2" in the "Math supplement A" notes):

Method 2: systematic method of solving:

$$\ddot{X} \equiv \frac{d^2X}{dt^2} = 0$$

$$\Rightarrow \int_0^{t_f} \frac{d^2X}{dt^2}\, dt = \int_0^{t_f} 0 \, dt$$

$$\Rightarrow \left. \frac{dX}{dt} \right|_0^{t_f} = 0$$

$$\Rightarrow \frac{dX(t=t_f)}{dt} - \frac{dX(t=0)}{dt} = 0$$

• Now, but we can choose $t_f$ to be any value we want. So drop the subscript "f" from $t_f$ and write "t" instead ($t_f$ is an independent variable)

• $\frac{dX}{dt}(t=0)$ is a constant (some number: In particular, value of $\frac{dX}{dt}$ at $t=0$)

Let's call $\frac{dX(t=0)}{dt} \equiv V_0$.

↑ some constant.

over →

Then we have:

$$\frac{d\vec{X}(t)}{dt} = V_0$$

$$\Rightarrow \int_0^{t_f} \frac{dX}{dt}\, dt = \int_0^{t_f} V_0\, dt$$

$$\Rightarrow \underbrace{\vec{X}(t=t_f)}_{\overset{\shortparallel}{\vec{X}(t)}} - \underbrace{\vec{X}(0)}_{\text{call this } X_0} = V_0 t_f - 0.$$

Again, drop the subscript "f"

Thus: we have;

$$\boxed{\vec{X}(t) = V_0 t + X_0}$$

← sol'n to $\ddot{X} = 0$

↑ 2nd order differential eq'n. And indeed, we see 2 free parameters $V_0$ and $X_0$.

Recall that $\vec{X}(t) = X_1(t) + X_2(t)$

- What motion does $\vec{X}(t)$ describe?

- Recall from your high school physics that center of mass position is: in our particular example:

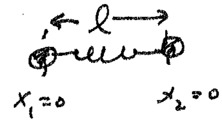
$$X_{com} = \frac{mX_1 + m(X_2 + l)}{m+m}$$

$l \equiv$ rest length of spring

$$\Rightarrow X_{com} = \frac{X_1 + (X_2 + l)}{2}$$

so: $\vec{X} = 2X_{com} - l$

$l \equiv$ initial rest length of spring.




$\vec{X}(t)$ is thus describing (up to factor 2) the position of the center of mass of particle; $\Rightarrow \dfrac{\vec{X} + l}{2} = \vec{X}_{com}(t)$

$$\Rightarrow \boxed{\vec{X}_{com}(t) = \frac{V_0}{2} t + \frac{X_0 + l}{2}}$$

Hence: Velocity of COM is $\frac{V_0}{2}$ if the motion is indeed described by $\ddot{X} = 0$.

- Going back to the 2 EOMs (① & ②) on (pg 55):
  - Subtract Eqn ② from ① to get:

①−②: $m(\ddot{x}_1 - \ddot{x}_2) = +k(x_2 - x_1) + k(x_2 - x_1)$
$= -2k(x_1 - x_2)$

$$\Rightarrow \boxed{\underbrace{(\ddot{x}_1 - \ddot{x}_2)}_{\ddot{Y}} + 2\omega_0^2 \underbrace{(x_1 - x_2)}_{Y} = 0}$$ Where $\omega_0 = \sqrt{\frac{k}{m}}$

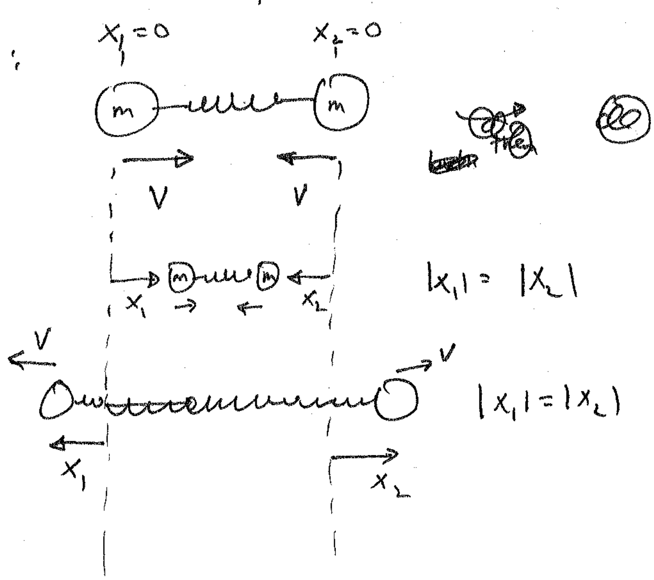
where we defined $\boxed{Y(t) \equiv x_1(t) - x_2(t)}$

SHM !!
(so, $Y(t)$ behaves like SHO.) with angular frequency $\sqrt{2}\,\omega_0$.

$$\Rightarrow \boxed{Y(t) = C\cos(\sqrt{2}\,\omega_0 t - \phi)}$$

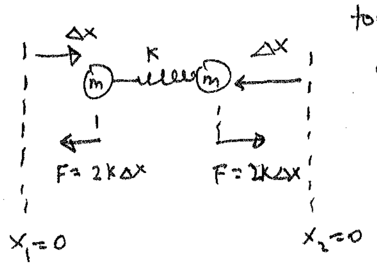
[ What motion, in terms of 2 blocks, does this describe?

Ans:




Ans: As shown in figure, $Y(t)$ is describing the 2 particles moving to, and away from, each other at the same speed;
(i.e. Particles 1 & 2 are mirror images of one another.)

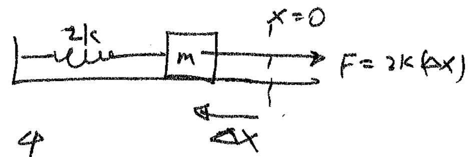
To see this, notice that if both particles move in towards each other by distance $\Delta x$ from their equilibrium positions $\left(\begin{smallmatrix} x_1 = 0 \\ x_2 = 0 \end{smallmatrix}\right)$ then spring is compressed by $(2\Delta x)$.
So Each particle experiences force $(2k)(\Delta x)$.




But as far as just one of the particles is concerned, that particle moved by distance $\Delta x$ and as a result, experiences restoring force of $(2k)\Delta x$.

This is as if that single particle is attached to a spring with Hooke's constant of $2k$ (instead of $k$).

~~2k as spring~~ i.e.




For this system, Eqn is: $\ddot{x} + \left(\frac{2k}{m}\right)x = 0$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{2k}{m}} = \sqrt{2}\,\omega_0}$$

And indeed, $\omega = \sqrt{2}\,\omega_0$ is the angular frequency of $Y(t)$.

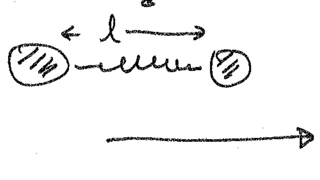
$\omega_0 = \sqrt{k/m}$

Summary:

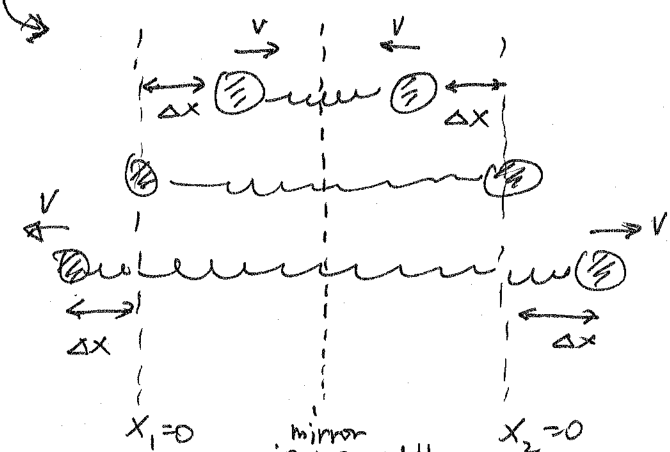
To recap:

$$\boxed{X(t) = V_0 t + X_0}$$

$$\boxed{Y(t) = C\cos(\sqrt{2}\,\omega_0 t - \phi)}$$




(Both particles maintain a fixed distance (unstretched spring) and move w/ velocity $V_0$)




(particles 1 & 2 are mirror images of each other).

Claim: All other types of motion that these 2 particles can execute, ~~are~~ can be described by some combination (addition & subtraction only) of motion described by $X(t)$ and motion described by $Y(t)$.

We'll see this in next class.