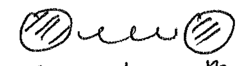
Continuing the problem from last class: [~~Tu~~ Wed. July 2, 08]

(see pg 55):

We found: $X(t) \equiv x_1 + x_2 = V_0 t + X_0$
and $Y(t) = C \cos(\sqrt{2}\,\omega_0 t - \phi)$ $\Big\}$ describing ↯

$Y \neq X_1 - X_2$

(where $\omega_0 \equiv \sqrt{k/m}$.)



~~But~~ our real goal is to find what $x_1(t)$ and $x_2(t)$ are:

and since we know $X = x_1 + x_2$ and $Y = x_1 - x_2$,

we get: $\boxed{x_1(t) = \dfrac{X+Y}{2}}$ ~~...~~

and, $\boxed{x_2(t) = \dfrac{X-Y}{2}}$

$\Rightarrow \quad x_1(t) = \dfrac{V_0}{2} t + \dfrac{X_0}{2} + \dfrac{C}{2} \cos(\sqrt{2}\,\omega_0 t - \phi)$

(And from pg 57: $\dfrac{V_0}{2} = V_{com}$ ← uniform velocity of COM.

(And to avoid constantly writing "$\frac{1}{2}$", we instead define $\dfrac{X_0}{2} \equiv \tilde{X}_0$ and $\dfrac{C}{2} \equiv \tilde{C}$)

So $\boxed{x_1(t) = V_{com} t + \tilde{X}_0 + \tilde{C} \cos(\sqrt{2}\,\omega_0 t - \phi)}$

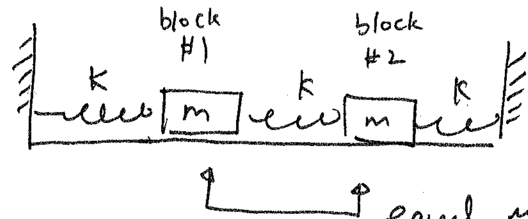
and $\boxed{x_2(t) = V_{com} t + \tilde{X}_0 - \tilde{C} \cos(\sqrt{2}\,\omega_0 t - \phi)}$

↑ General solution to original EOMs

Another example, but this time, using "matrix" approach

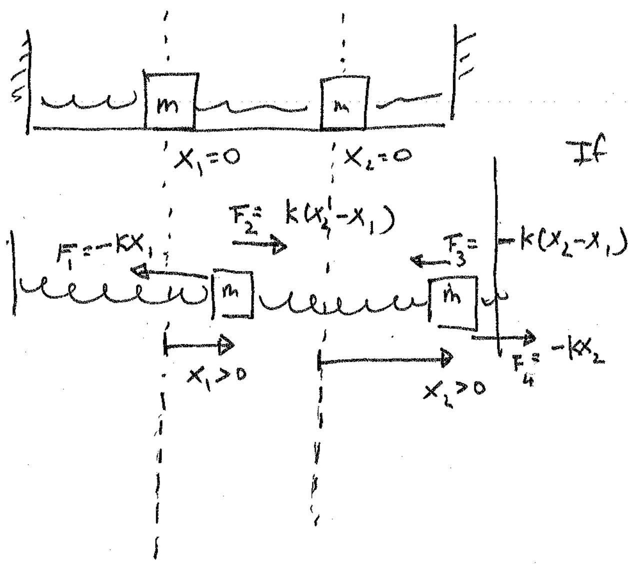
(Be sure to review your matrix addition, multiplication rules, etc.)

Ex



equal masses (No friction).

① Derive EOMs for the 2 blocks



If $x_2 > x_1$: then $(x_2-x_1)>0$ and this denotes the case in which the middle spring connecting the 2 blocks is stretched.

Similarly: if $(x_2-x_1)<0$, then middle spring is compressed.

Finally, if $x_2=x_1$; $\Rightarrow$ $x_2-x_1=0$

(middle spring is unstretched/compressed.)

So, EOM for block #1 :

$$m\ddot{x}_1 = F_1+F_2 = -kx_1+k(x_2-x_1) \Rightarrow \boxed{\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0}$$

... EOM ①

EOM for block #2 :

$$m\ddot{x}_2 = F_3+F_4 = -kx_2 - k(x_2-x_1) \Rightarrow \boxed{\ddot{x}_2 + 2\omega_0^2 x_2 - \omega_0^2 x_1 = 0}$$

... EOM ②

Where we define $\omega_0 = \sqrt{k/m}$

②
Next, solve EOMs, but first, ~~rearrange~~ write the 2 EOMs in matrix notation,

over.

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

← EOMs written in matrix form. Need to determine the actual values of a, b, c, d.

We want
$$ax_1 + bx_2 = -2\omega_0^2 x_1 + \omega_0^2 x_2$$
$$cx_1 + dx_2 = -2\omega_0^2 x_2 + \omega_0^2 x_1$$

$\Rightarrow$ $\underline{a = -2\omega_0^2}$, $\underline{b = \omega_0^2}$, $\underline{c = +\omega_0^2}$, $\underline{d = -2\omega_0^2}$.

So:
$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(3) Guess and solve: Let's look at the $\mathbb{C}$-equivalent EOM (also in matrix form), guess its solution, and check. Then at the end, we'll take the real part of the solution. That will be our $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

$\therefore$ $\mathbb{C}$-equivalent EOM: $x \rightarrow z$

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Guess: $\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$; where A & B are some constants and α = yet to be determined parameter.

<u>Check</u>: First, note that $\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = -\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$

So: EOM become:

$$-\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$$

$$\Rightarrow \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$$

↑ used the fact that

$$-\alpha^2 = -\alpha^2 \underset{\substack{\uparrow \\ \text{identity} \\ \text{matrix}}}{\mathbb{1}} = -\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix} + \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{pmatrix} \right\} \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t} = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2\omega_0^2 - \alpha^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \alpha^2 \end{bmatrix}}_{\text{↑ call this matrix "M"}} \begin{bmatrix} A \\ B \end{bmatrix} \cancel{e^{i\alpha t}} = 0 / e^{i\alpha t} = 0 .$$

So, we have

$$\begin{bmatrix} 2\omega_0^2 - \alpha^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \alpha^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \leftarrow \boxed{\text{Eq'n (1)}}$$

Eq'n (1) : $M \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

* For a moment, let's suppose that M has an inverse M^{-1}.
That is, there exists a matrix (called "M^{-1}") such that

$$M \cdot M^{-1} = M^{-1} \cdot M = \underbrace{\mathbb{1}}_{\text{"Identity" matrix (like "1")}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Notice that not all matrices have ~~things~~ such an inverse matrix:
For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ times any 2×2 matrix yields zero, not $\mathbb{1}$.
Thus, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse matrix.)

Going back to our original question, if M has an inverse, then:

$$M^{-1} \cdot M \begin{pmatrix} A \\ B \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ ← we can multiply both sides of Eq'n (1) with M^{-1}.

$$\Rightarrow \underbrace{(M^{-1} \cdot M)}_{= \mathbb{1}} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\Rightarrow \mathbb{1} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = 0$$ ⇒ A and B are <u>both</u> <u>simultaneously</u> <u>zero</u>.

But this is a <u>trivial</u> <u>solution</u>.

Since we now have $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} e^{i\alpha t} = 0$.

⇒ Both $z_1(t) = 0$ and $z_2(t) = 0$

and hence their real parts: $\boxed{x_1(t) = 0 \quad x_2(t) = 0.}$

This is merely describing the situation in which both blocks are at rest, in their equilibrium positions. This is indeed a perfectly allowed motion (thus a solution to our EOMs)

(This is what it means to be a solution to EOM)

but this case happens to be uninteresting!

. Hence, it follows that we need to look for situations in which the matrix M has no inverse. (since then our argument on pg64 doesn't apply.)

· So, when is "M" not invertible? (i.e. has no inverse?)

· Though I don't give a rigorous proof here, it turns out that the inverse of any matrix M (i.e. M can be 2×2, 3×3, 4×4, ... $N\times N$ matrix)

is
$$\boxed{M^{-1} = \frac{1}{\text{Det}(M)} \times \left(\begin{array}{l}\text{some matrix you} \\ \text{get by doing some} \\ \text{matrix operations on } M\end{array}\right)}$$

where $\boxed{\text{Det}(M) = \text{determinant of matrix } M.}$

& some #.

So, if ~~det~~ $\text{Det}(M) = 0$, then we'd be dividing by zero, ~~and~~ hence above equation doesn't hold. $\Rightarrow$ $\boxed{M^{-1} \text{ does not exist as long as } \text{Det}(M) = 0}$

· For a 2×2 matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$; $\boxed{\text{Det}(M) = m_{11}m_{22} - m_{12}m_{21}}$

Going back to our original question,

$$M = \begin{bmatrix} 2\omega_0^2 - \alpha^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \alpha^2 \end{bmatrix} \quad \leftarrow$$ See pg 63

So, Notice that <u>we have not yet defined what "α" is</u>.

so, to make $\mathrm{Det}(M) = 0$, we will pick just the right values of α so that $\mathrm{Det}(M) = 0$.

<u>Let's do this</u>: We want:

$$0 = \mathrm{Det}(M) = (2\omega_0^2 - \alpha^2)^2 - \omega_0^4 \quad \leftarrow \text{using formula on pg 65}$$

$$\Rightarrow \omega_0^4 = (2\omega_0^2 - \alpha^2)^2$$

$$\Rightarrow \pm\omega_0^2 = 2\omega_0^2 - \alpha^2$$

$$\Rightarrow \alpha^2 = 2\omega_0^2 \mp \omega_0^2$$

① $\boxed{\alpha_1 = \pm\omega_0}$

② $\boxed{\alpha_2 = \pm\sqrt{3}\,\omega_0}$

(4 possible value of α)

So:

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$$

$$V_1 \equiv \begin{pmatrix} A \\ B \end{pmatrix} e^{i\omega_0 t} \qquad V_2 \equiv \begin{pmatrix} A \\ B \end{pmatrix} e^{-i\omega_0 t} \qquad V_3 \equiv \begin{pmatrix} A \\ B \end{pmatrix} e^{i\sqrt{3}\,\omega_0 t} \qquad V_4 \equiv \begin{pmatrix} A \\ B \end{pmatrix} e^{-i\sqrt{3}\,\omega_0 t}$$

← Define $V_{1,2,3,4}$.

Now, notice that: $(\alpha = \omega_0 \text{ here.})$:

$$M \cdot V_1 = 0 \Rightarrow \begin{pmatrix} 2\omega_0^2 - \alpha^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underset{\alpha = \omega_0}{\Rightarrow} \begin{pmatrix} \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \omega_0^2 A - \omega_0^2 B \\ -\omega_0^2 A + \omega_0^2 B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{A = B}$$

(so A & B are not independent of each other!!)

Thus: $V_1(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t}$

Also, $V_2(t) = A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t}$

A_1 & A_2 are independent free parameters

Similarly: for $\alpha_2 = \pm\sqrt{3}\,\omega_0$:

$$M \cdot V_{3,4} = 0 \Rightarrow \begin{pmatrix} 2\omega_0^2 - \alpha^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(plug in $\alpha = \pm\sqrt{3}\,\omega_0$)

$$\Rightarrow \begin{pmatrix} -\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \omega_0^2 \begin{pmatrix} A + B \\ A + B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow A = -B$$

So for V_3 & V_4, A & B not independent of each other!

Thus, $\vec{V}_3(t) = A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t}$

$\vec{V}_4(t) = A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}$

A_3 & A_4 are free parameters

~~Thus~~ And so the following are solutions of $\mathbb{C}$-equivalent EOM:

$$\left\{ \begin{aligned} \vec{V}_1(t) &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} \\ \vec{V}_2(t) &= A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t} \\ \vec{V}_3(t) &= A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t} \\ \vec{V}_4(t) &= A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t} \end{aligned} \right\}$$

But what's the general solution?

- Use linearity (superposition principle) again!

Notice that

distributive law for Matrix multiplication

$$M \cdot (\underbrace{\vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4}_{\vec{V}_{general}}) = \underbrace{M \cdot \vec{V}_1}_{0} + \underbrace{M\vec{V}_2}_{0} + \underbrace{M\vec{V}_3}_{0} + \underbrace{M\vec{V}_4}_{0} = 0$$

$\therefore$ $\vec{V}_{general}(t)$ is also a solution. And in fact, since it contains 4 free parameters: A_1, A_2, A_3, A_4; it's the most general sol'n to EOM. $\left(\begin{matrix} \text{2 free parameter} \\ \text{per particle} \end{matrix} \right) \times$ 2 particles $=$ 4 free params.

$\therefore$ The most general solution to $\mathbb{C}$-equivalent eoms:

$$\boxed{\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} + A_2 \begin{pmatrix} 1 \\ +1 \end{pmatrix} e^{-i\omega_0 t} + A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}}$$

To get the actual, physically meaningful part of answer: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, we need to extract the real part of $\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$.

But by now, we can just "read" it off / recognize the real part from $z_{1,2}(t)$:

① $A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t}$

$$= \begin{bmatrix} A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} \\ A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} \end{bmatrix} \xrightarrow[\text{part}]{\text{real}} \begin{bmatrix} C_1 \cos(\omega_0 t - \phi_1) \\ C_1 \cos(\omega_0 t - \phi_1) \end{bmatrix}$$

② $A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}$

$$= \begin{bmatrix} A_3 e^{i\sqrt{3}\omega_0 t} + A_4 e^{-i\sqrt{3}\omega_0 t} \\ -A_3 e^{i\sqrt{3}\omega_0 t} - A_4 e^{-i\sqrt{3}\omega_0 t} \end{bmatrix} \xrightarrow[\text{part}]{\text{real}} \begin{bmatrix} C_2 \cos(\sqrt{3}\omega_0 t - \phi_2) \\ -C_2 \cos(\sqrt{3}\omega_0 t - \phi_2) \end{bmatrix}$$

Thus: the real, physically meaningful, general sol'n to EOM:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\,\omega_0 t - \phi_2)}$$

↑ Final answer: General sol'n to EOM.

Note: 4 free parameters:

C_1, C_2, ϕ_1, ϕ_2.

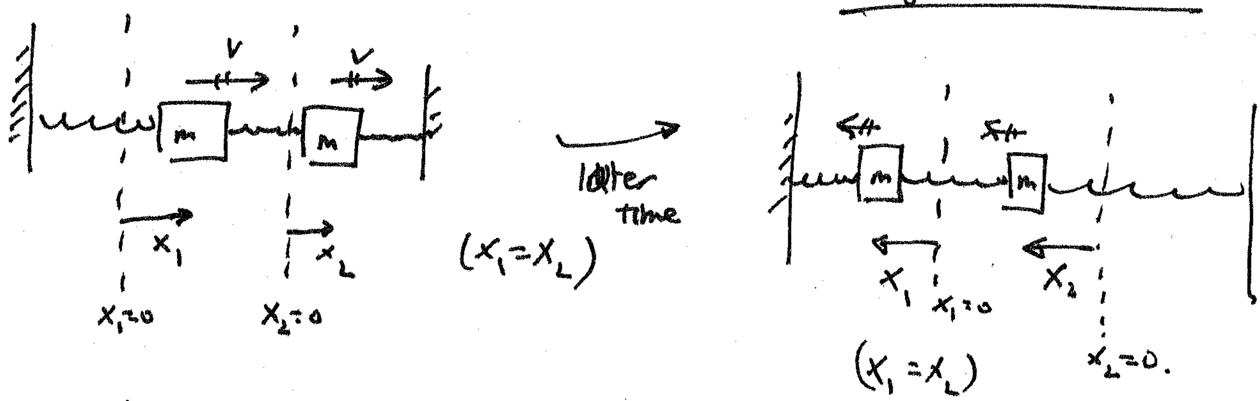
physical interpretation:

Notice that $[C_1 \cos(\omega_0 t - \phi_1)] \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ describes a normal mode

in which both blocks are oscillating in phase w/ each other.

w/ angular frequency ω_0:

i.e.



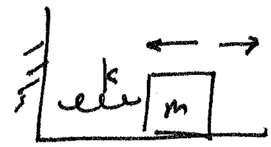
* The middle spring always remains at its rest length

$(x_2 - x_1 = 0)$

So one block doesn't know the presence of the other block

(the only way for block #1 to feel the presence of block #2 is if the middle spring connecting the two is stretched or compressed. But since $x_2 - x_1 = 0$ at all times, ($\because x_1(t) = x_2(t)$) they can't feel each other.)

so as far as block #1 alone (or block #2 alone) is concerned, ~~its~~ it's equivalent to :

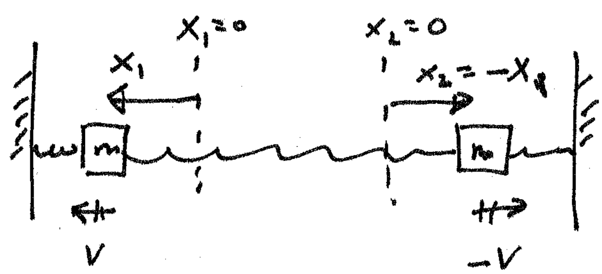


$\Rightarrow$ oscillates w/ angular freq. $\omega_0 = \sqrt{k/m}$.

Next, notice that $C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\,\omega_0 t - \phi_2)$ describes a normal mode in which the 2 blocks are oscillating out of phase with respect to each other, w/ "normal angular frequency" $\boxed{\sqrt{3}\,\omega_0}$.

this is because $X_1(t) = -X_2(t)$ at all times.

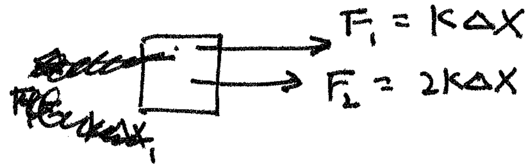
$\Rightarrow$



So, if ~~the~~ particle #1 moves to the left by ΔX_1 ~~from~~ (from initially $X_1 = 0$), the left spring compresses by ΔX_1. But by this time, block #2 has moved to the right by ΔX_1 ($\because X_1(t) = -X_2(t)$).
"because"

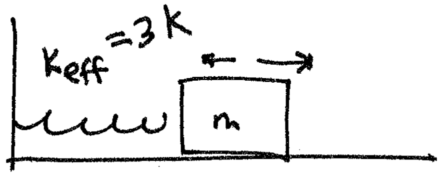
so, the middle spring has now been stretched by $2\Delta X_1$.

so, the ~~net force~~ free body diagram of block #1 is :



$\Rightarrow F_1 + F_2 = F_{net} = 3K(\Delta X)$.

So, this is equivalent to the following system:



whose natural angular frequency is $\sqrt{\frac{k_{eff}}{m}} = \sqrt{3}\sqrt{\frac{k}{m}} = \boxed{\sqrt{3}\,\omega_0}$.

This is indeed what our normal mode angular frequency is!

<u>Some Language / terms</u>: The 2 "α"'s we found:

(ignoring sign difference)

$\alpha_1 = \pm\omega_0$ and $\alpha_2 = \pm\sqrt{3}\,\omega_0$ are called <u>normal mode angular frequencies</u>.

- The motion that each of these 2 describe are called "<u>normal modes</u>".

- <u>Normal coordinates</u> are: $\boxed{\begin{pmatrix}1\\-1\end{pmatrix} \text{ and } \begin{pmatrix}1\\-1\end{pmatrix}.} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$.