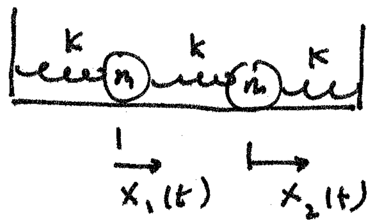
(P073)

Wed. July 9, 08

· Normal modes:



Last time, we solved the EOMs of this system and found:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\,\omega_0 t - \phi_2)}$$

* The normal coordinates are: $V_1 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv V_2$ ← call these 2 vectors V_1 and V_2.

* Corresponding normal angular frequencies are: $\omega_0 \equiv \omega_1$ $\quad \sqrt{3}\,\omega_0 \equiv \omega_2$ ← call these 2 ω_1 and ω_2

* The 2 normal modes are: $C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_1)$ (symmetric motion)

and $C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\,\omega_0 t - \phi_2)$ (Anti-symmetric motion.)

* Recall that we obtained above solutions by deriving the EOMs, then writing ~~the~~ it in matrix form, (the C-# equivalent), and obtained a matrix eq'n that looked like:

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

a, b, c, d are some $\mathbb{R}$ numbers.

Then by guessing: $\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$ (where α will turn out to be the normal angular frequencies)

we got: $$\boxed{-\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \begin{pmatrix} A \\ B \end{pmatrix}}$$

A, B ← constants.

Then we found $\alpha_{1,\pm} = \pm\omega_1$ and $\alpha_{2,\pm} = \pm\omega_2$.

So the boxed eqn on previous page becomes:

$$\boxed{-\omega_1^2 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \leftarrow \text{for normal mode : } \omega_1}$$

$$\boxed{\text{and } -\omega_2^2 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \leftarrow \text{for normal mode : } \omega_2.}$$

(A, B are #'s) ~~then you~~

Then solving for A & B in each of the 2 matrix eqns shown above, we get: $A = B$ (for normal mode: ω_1)

and $A = -B$ (for normal mode: ω_2).

All of this is just a recap of what we did last (~~Wed.~~) class.

- Today, we want to get a deeper understanding of normal modes.

- To motivate this, notice that the 2 normal coordinates in this problem: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are orthogonal vectors.

That is, taking the dot product of V_1 & V_2:

Dot product: $V_1 \cdot V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot 1 + (1)\cdot(-1) = 1 - 1 = 0.$

$\Downarrow$

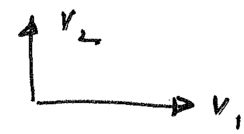
so V_1 is indeed perpendicular to V_2.

As we will learn, when we study waves in depth, any general, arbitrary motion executed by coupled oscillators can be described by a <u>superposition of normal modes</u>.

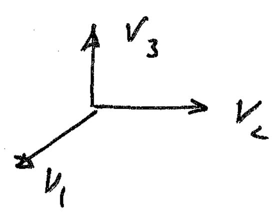
The idea is that any 2 normal coordinates (say V_1, V_2) are orthogonal to each other, just like basis vectors.

(e.g. If we have a coupled oscillation of N particles, we'd have N normal mode, described by N normal coordinates $V_1, V_2, V_3, \ldots, V_N$)
Then, $V_i \cdot V_j = 0$ for any pair (i, j) $(i \neq j)$
$1 \leq i, j \leq N$)

Recall that in 2Dimensions, you need 2 orthogonal (perpendicular) vectors to describe all vectors in 2D plane:



In 3Dimensional space, you need 3 orthogonal (perpendicular) vectors to describe all vectors in 3D plane:



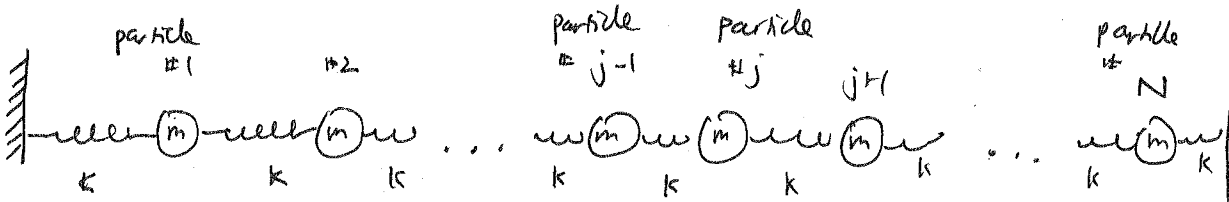
Similarly, for N particles in a coupled oscillator system, you need N orthogonal "basis states of motion" (what we call normal modes) w/ normal coordinates $V_1, V_2, \ldots, V_N$ to describe ~~all~~ any arbitrary motion of these N particles as a linear combination of these N basis vectors.

Again,
This will become clearer when we study <u>Fourier Series</u> in waves.

Coupled oscillations of many particles

So far, we studied oscillation of one, or coupled oscillations of 2 or 3 particles. What happens if we have a coupled oscillations of $\underline{N}$ particles where N is very large?

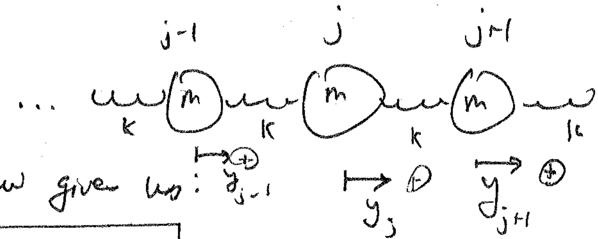
EX:



Goal: Find the equation of motion (EOM) for this system. (Notice that there are N EOMs!! (one for each particle)

To do so, focus on the j^{th} particle and find its EOM:

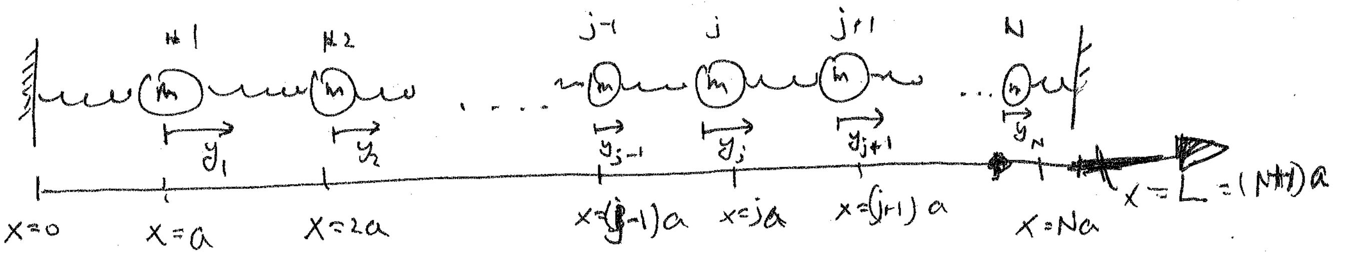
- For the j^{th} particle:



Applying Newton's 2^{nd} law gives us:

$$\boxed{m\ddot{y}_j = -k(y_j - y_{j-1}) + k(y_{j+1} - y_j)}$$ ... EOM for j^{th} particle

Coordinate system we're using is:



x = distance (position) measured from left wall.

$a \equiv$ Equilibrium (rest) length of spring $\Rightarrow$ a is thus the separation distance between 2 adjacent particles.

$y_j \equiv$ $\underline{\text{displacement}}$ of j-th particle from its equilibrium position.

$j = 1, 2, \dots, N.$ $\quad N \equiv$ total # of particles

$L = (N+1)a$ ← Length of the entire spring-atom system. (Distance between the 2 walls.)

so, we've found EOM for j^{th} particle $(j=1, \ldots, N)$

$\Rightarrow$ We've effectively found EOM for all N particles.

Notice that typically N is a very large #. (N ~~~~ Avogadro # ~ 10^{23} particles).

Finding the solution to 10^{23} equations is neither illuminating nor technically feasible in many cases.

(i.e. Solving the system of 10^{23} EOMs means we will be able to write down

$$\begin{cases} y_1(t) = \text{some function of } t & \leftarrow \text{motion of particle \#1} \\ y_2(t) = \text{some function of } t & \leftarrow \text{motion of particle \#2} \\ \vdots \qquad \vdots & \vdots \\ y_N(t) = \text{some function of } t & \leftarrow \text{motion of particle \#N} \end{cases}$$

But, when we study objects made up of N atoms / smaller objects, what we're really concerned with is not the behavior of each of the individual atoms / smaller objects making up the object, but ~~two~~ rather we're interested in how the macroscopic object behaves as a result of collective motion / behavior of all N particles.

~~one~~ Idea: Perhaps we can turn the N eq'ns (N EOMs) all into a single eq'n that ~~describes~~ describes the collective ~~motion~~ oscillations of all ~~of~~ N ~~two~~ particles. $\leftarrow$ This will turn out to be "wave eq'n".

$\uparrow$ we can do this for large N and small "a".

(N = # particles, a = ~~equilibrium~~ separation distance between adjacent particles in equilibrium.)

So, going back to the example on Pg 76 :

we had EOM of j^{th} particle: $m\ddot{y}_j = -k(y_j - y_{j-1}) + k(y_{j+1} - y_j)$

Now, N is very large. (On the order of Avogadro # .)
(m is pretty small (mass of single atom, let's say.)).
Then m is hard to measure, but $Nm \equiv M$ ← total mass of ~~slinky~~ slinky is easy to measure. (Using a scale in your bathroom, for example.)
We want to express our equations in terms of quantities we can easily measure. So, let's replace "m" in above eqn with M by:

$$m = \frac{M}{N}$$

; but N is also typically difficult to measure (i.e. Not easy to count the large # of atoms making up slinky.)

But notice that:

$$m = \frac{Ma}{Na} = \left(\frac{M}{Na}\right) a$$

$$= \left(\frac{M}{L}\right) a \quad \leftarrow L = (N+1)a \approx Na \quad (\because N >> 1)$$

$$= \rho a$$

where ρ = mass density / length.
↑ easy to measure.

So: EOM becomes:

$$\rho a \ddot{y}_j = -k(y_j - y_{j-1}) + k(y_{j+1} - y_j)$$

$$\Rightarrow \ddot{y}_j = \frac{+k}{\rho} \left\{ \frac{(y_{j+1} - y_j)}{a} - \frac{(y_j - y_{j-1})}{a} \right\}$$

But, L is fixed while N large $\Rightarrow \frac{L}{N+1} \underset{(N>>1)}{\approx} \frac{L}{N} = a$

↑ v. small since N large.

This is a bit subtle: what we're saying is that N is sufficiently large so that a is very small. ~~(continuum limit)~~.

[called: continuum limit]

So, $a \equiv \delta x$ ← v. small.

⇒ particles j and $j+1$ are very close to each other.

And we expect y_{j+1} and y_j to be very close to each other as well.

(i.e. $y_{j+1} - y_j = \delta y$.) (we're examining only <u>small</u> vibrations in this problem)

So: $$\frac{y_{j+1} - y_j}{a} \approx \left.\frac{\delta y}{\delta x}\right|_{x=ja}$$ means "$\frac{\delta y}{\delta x}$ evaluated at $x=ja$".

and $$\frac{y_j - y_{j-1}}{a} \approx \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a}.$$

so EOM becomes:

$$\ddot{y}_j \approx \frac{K}{\rho}\left\{\left.\frac{\delta y}{\delta x}\right|_{x=ja} - \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a}\right\}$$

Now, from Pset #1, we learned that when N identical springs are connected in series, the effective spring constant of the conglomerate spring is $K_{eff} = \frac{K}{N}$.

Now, assuming that rest length of spring is small compared to L,

$K_{eff} L \approx$ force felt by both ends of the wall. ← something we can easily measure.

So let's write EOM in terms of this force.

$$\Rightarrow \quad K_{eff} L = \frac{K}{N} L = \frac{K}{N} Na = Ka.$$

$$\therefore \quad \ddot{y}_j = \frac{K}{\rho}\left\{\left.\frac{\delta y}{\delta x}\right|_{x=ja} - \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a}\right\}$$

$$\Rightarrow \quad \ddot{y}_j = \frac{(Ka)}{\rho}\cdot\frac{1}{a}\left\{\left.\frac{\delta y}{\delta x}\right|_{x=ja} - \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a}\right\}$$

force felt by both ends of wall.

But
$$\frac{\left\{ \left.\frac{\delta y}{\delta x}\right|_{x=ja} - \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a} \right\}}{a} \underset{(a\ \text{small})}{=} \frac{\left.\frac{\delta y}{\delta x}\right|_{x=ja} - \left.\frac{\delta y}{\delta x}\right|_{x=(j-1)a}}{\delta x}$$

Notice that this is, by definition, the 2nd derivative of y with respect to x.

$$\approx \left.\frac{\delta^2 y}{\delta x^2}\right|_{x=ja} \quad \leftarrow \text{means "evaluated" at } x=ja$$

~~By~~ definition.

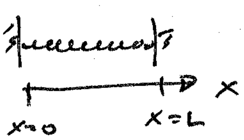
$$\equiv \frac{\partial^2 y(x=ja, t)}{\partial x^2}$$

"by definition".

So, EOM becomes:

$$\boxed{\frac{\partial^2 y(x=ja,t)}{\partial t^2} = \frac{(ka)}{\rho}\frac{\partial^2 y(x=ja,t)}{\partial x^2}}$$

This should hold for <u>any</u> j. And in fact since N is large and "a" is v. small, atoms are really closely packed. So we can treat "x" to be a <u>continuous</u> variable. (position from the wall.)



$\therefore$ Drop the "ja" and write:

$$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2} = \left(\frac{ka}{\rho}\right)\frac{\partial^2 y(x,t)}{\partial x^2}} \leftarrow \text{"wave" eq}^n.$$