Fri. July 11, 08

The boxed eqn at the bottom of previous pg. is the single eqn describing the collective motion (oscillation) of all N particles.

This is exactly what we wanted to get. (see pg 77)

Notice that the dimension of $\frac{ka}{\rho}$ is: $\left[\frac{ka}{\rho}\right] = \frac{[k][a]}{[\rho]} = \frac{\frac{\text{mass}}{\text{Length}} \cdot \frac{\text{Length}}{\text{time}^2} \cdot \text{Length}}{\text{mass}/\text{Length}}$

$= \left(\text{Length}/\text{time}\right)^2$

$= \text{speed}^2$.

so let $V = ka/\rho$
speed.

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 y}{\partial x^2}}$$

← "wave" eqn.
we haven't proved that this is actually a wave eqn yet!! (Thus the quotation marks.)

we will soon see that V is the wave speed.

Now, let's solve above eqn: i.e. What is $y(x, t) = ?$

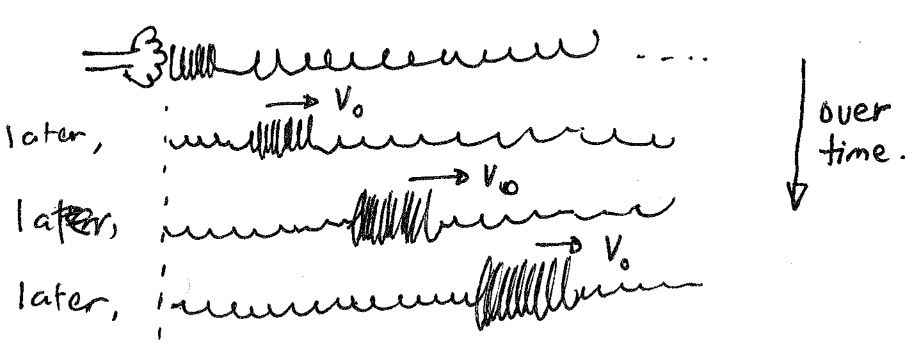
(Note: Louis derived an eqn that looks like ~~the~~ the one we have boxed above in his recitation, for a string carrying transverse wave. So we have every right to suspect that above eqn is indeed describing wave!).

By solving the eqn, we will prove that indeed, above eqn is ~~describing~~ describing a wave moving w/ speed V.

We solve by guessing what $y(x, t)$ is.

To do This, let's motivate our guess from physical reasoning.

Going back to our slinky example, if you hold one end of slinky and let it go;

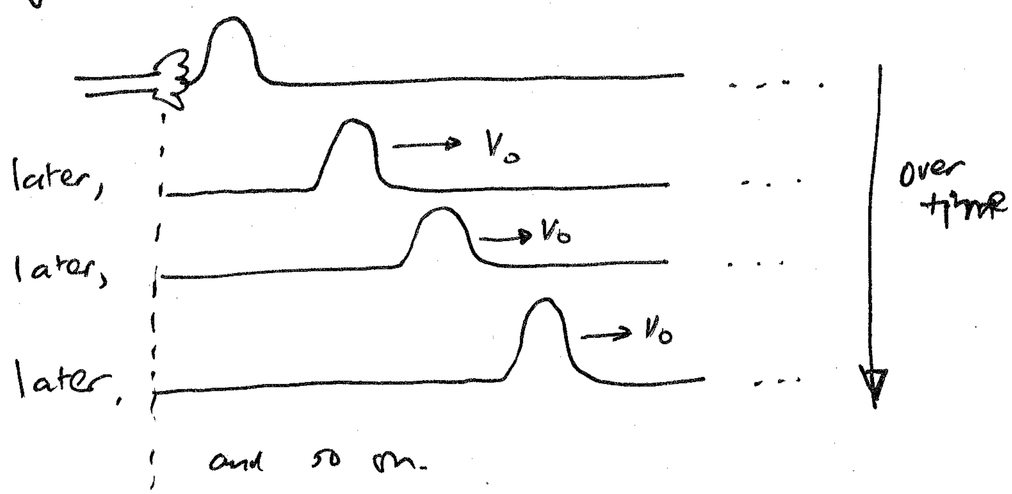


and so on.

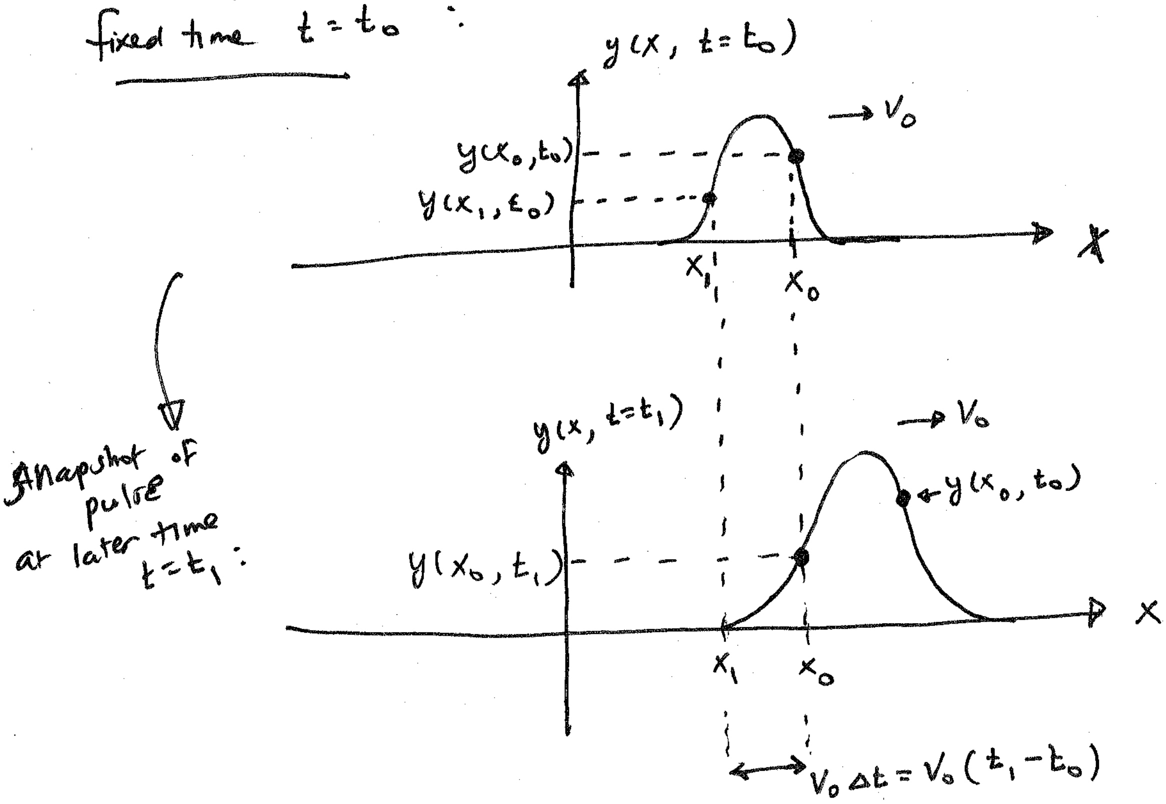
the disturbance you created at one end of slinky with your hand travels down along the slinky with some speed V_0.

~~Graphing~~ Since this is clearly a possible motion supported by the collective motion of N particles making up the slinky, it must be one of the solutions of the "wave" eq'n we derived.

Similarly, in Louis' example of a string, hold one end of string, shake it up & down once, and watch a pulse travel down the string : (w/ speed V_0).



Graphing the height of pulse $y(x,t)$ with respect to x at a fixed time $t = t_0$:



Thus, $y(x_0, t_1) = y(x_1, t_0)$ but $x_1 = x_0 - V_0(t_1 - t_0)$

$= y(x_0 - V_0(t_1 - t_0), t_0)$

Now, we can pick $t_0 = 0$. This doesn't change anything ~~about~~ in our argument.

So we have:

$$y(x_0, t_1) = y(x_0 - V_0 t_1, 0)$$

· And we can pick t_1 to be any later time. ⇒ t_1 is an independent variable.

Drop the subscript "1" and write "t" instead of "t_1".

· Similarly, we can pick x_0 to be anything.

→ Drop subscript "0" and write "x" instead of "x_0".

⇒ we have: $$\boxed{y(x,t) = y(x - V_0 t, 0)}$$

so, motivated by this physical reasoning, let's try to plug in our guess:

> Guess: $y(x,t)$ is an arbitrary function, which has the property that $y(x,t) = y(x - V_0 t, 0)$
>
> ~~(where we have yet to specify)~~

Plugging this arbitrary guess into our "wave eq'n":

(Does our guess satisfy the wave eq'n?)

LHS of wave eq'n:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial y(x - V_0 t, 0)}{\partial t}\right)$$ ← using the property • $y(x,t) = y(x - V_0 t, 0)$

$$= \frac{\partial}{\partial t}\left(\frac{d y(z,0)}{dz} \cdot \frac{\partial z}{\partial t}\right)$$ ← chain rule: where $z \equiv x - V_0 t$.

$$= -V_0 \frac{\partial}{\partial t}\left(\frac{dy(z,0)}{dz}\right)$$

over →

continued ...

$$= -V_0 \frac{\partial}{\partial t}\left(\frac{dy(z,0)}{dz} \right)$$

$$= -V_0 \frac{d^2 y(z,0)}{dz^2} \cdot \frac{\partial z}{\partial t}$$

$$= V_0^2 \frac{d^2 y(z,0)}{dz^2} \quad \leftarrow \text{LHS of wave eqn.}$$

As for the RHS of wave eqn :

$$V^2 \frac{\partial^2 y(x,t)}{\partial x^2} = V^2 \frac{\partial^2 y(x - V_0 t, 0)}{\partial x^2} \quad \leftarrow \text{using the property: } y(x,t) = y(x - V_0 t, 0)$$

$$= V^2 \frac{\partial}{\partial x}\left[\frac{dy(z,0)}{dz} \cdot \underbrace{\frac{\partial z}{\partial x}}_{=1} \right]$$

$$= V^2 \frac{d^2 y(z,0)}{dz^2} \cdot \underbrace{\frac{\partial z}{\partial x}}_{=1}$$

$$= V^2 \frac{d^2 y(z,0)}{dz^2}$$

$\therefore$ LHS = RHS ; if and only if $V_0^2 \dfrac{d^2 y(z,0)}{dz^2} = V^2 \dfrac{d^2 y(z,0)}{dz^2}$

$$\Longleftrightarrow \boxed{V_0 = \pm V}$$

So, $y(x,t)$ with the property that $y(x,t) = y(x - Vt, 0)$ or $y(x,t) = y(x + Vt, 0)$ is a solution to the wave eqn.

~~wave~~ solution to wave eqn.

Notice that $V_0 = \pm V$ means that pulse can move either to the right ($V_0 = +V$) or to the left ($V_0 = -V$).

$$\Rightarrow \left[\begin{array}{ll} f_-(x,t) = g(x - vt) & (V_0 = +V) \leftarrow \text{pulse moving to the right} \\ f_+(x,t) = g(x+vt) & (V_0 = -V) \leftarrow \text{pulse moving to the left.} \end{array} \right.$$

Where, g is an arbitrary one variable (z) function ($g(z)$) that describes the shape of the pulse.

e.g. $g(z) = \sin(zk)$, or $g(z) = \cos(z\cancel{k})$ ← familiar sinusoidal wave (k = wave #)

$g(z) = e^{-z^2/a^2}$ ← Gaussian etc.

(a is some constant to make dimensions work out)

(~~where~~ So, as an example, if $g(z) = A\sin(kz)$ (A = constant, k = wave # ← Need this to make kz dimensionless

then plugging in $z = x - vt$;

$g(x-vt) = A\sin(k(x-vt))$.

And notice that

$$\frac{\partial g}{\partial x} = \frac{dg(z)}{dz} \cdot \frac{\partial z}{\partial x} \quad ; \quad \frac{\partial g}{\partial t} = \frac{dg(z)}{dz} \cdot \frac{\partial z}{\partial t}$$

chain rule.

or, if $g(z) = e^{-z^2/a^2}$,

then $f_-(x,t) = g(x-vt) = \exp\left[-\frac{(x-vt)^2}{a^2}\right]$.

The main point is that g can be ~~[illegible]~~ any arbitrary function; (since pulse shape can be arbitrary) as long as we plug in $z = x \mp vt$ and set $f_{\mp}(x,t) = g(x \mp vt)$, then f is a solution to wave eqn

Conclusion: We have just proved that $\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$ is an eqn that describes pulse (wave) of any shape moving with constant speed V.

- Linearity of wave eqn (superposition principle):

- Just like the SHM eqn we studied before midterm, the wave eqn obeys linearity (superposition) property.

To see this:

Consider
$$f(x,t) = f_-(x,t) + f_+(x,t)$$
$$= g(x-vt) + g(x+vt).$$

Then,
$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[g(x-vt) + g(x+vt)\right]$$
$$= \frac{\partial^2 g(x-vt)}{\partial t^2} + \frac{\partial^2 g(x+vt)}{\partial t^2}$$
$$= v^2 \frac{d^2 g(z)}{dz^2} + v^2 \frac{d^2 g(z)}{dz^2}$$

And
$$v^2 \frac{\partial^2 f}{\partial x^2} = v^2 \frac{\partial^2}{\partial x^2}\left\{g(x-vt) + g(x+vt)\right\}$$
$$= v^2 \left\{\frac{\partial^2 g(x-vt)}{\partial x^2} + \frac{\partial^2 g(x+vt)}{\partial x^2}\right\}$$
$$= v^2 \frac{d^2 g(z)}{dz^2} + v^2 \frac{d^2 g(z)}{dz^2}$$

Equal.
$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$
Wave eqn satisfied by $f(x,t) = f_+ + f_-$.

$\therefore$ If f_+, f_- are solutions, then so is $f = f_+ + f_-$

In fact, we can have 2 different shapes of pulses

(i.e. $g_1(z)$ and $g_2(z)$ $g_1 \neq g_2$.)

both of which are solutions to the wave eqn. (both travel at speed V.)

Then consider $\cancel{f(x,t) = g_1(x,t)}$

$$f(x,t) = f_1(x,t) + f_2(x,t)$$; where

$f_1(x,t) = g_1(x \mp Vt)$

$f_2(x,t) = g_2(x \mp Vt)$

then:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} + \frac{\partial^2 f_2}{\partial t^2}$$

$$= V^2 \left[\frac{d^2 g_1(z)}{dz^2} + \frac{d^2 g_2(z)}{dz^2} \right]$$

and

$$V^2 \frac{\partial^2 f}{\partial x^2} = V^2 \left[\frac{d^2 g_1(z)}{dz^2} + \frac{d^2 g_2(z)}{dz^2} \right].$$

Equal.

$\therefore$ $f(x,t) = f_1 + f_2$ is a solution to wave eqn as well.

(linearity.) shown here.

$\boxed{\text{So, we have just shown that the wave eqn is linear}}$

ok, so far, abstract since we've worked with a general (unspecified) shape of pulse $g(z)$.

Let's do a concrete example by picking a specific wave form: Sinusoidal Wave.

over

Ex: $f(x,t) = g(x - vt)$ ← pulse / wave moving to right w/ speed V

and let $g(z) = A \sin(kz)$

↑ $[k] = \frac{1}{\text{Length}}$ (so that kz is dimensionless inside the sine.)

$\therefore f(x,t) = g(x-vt)$

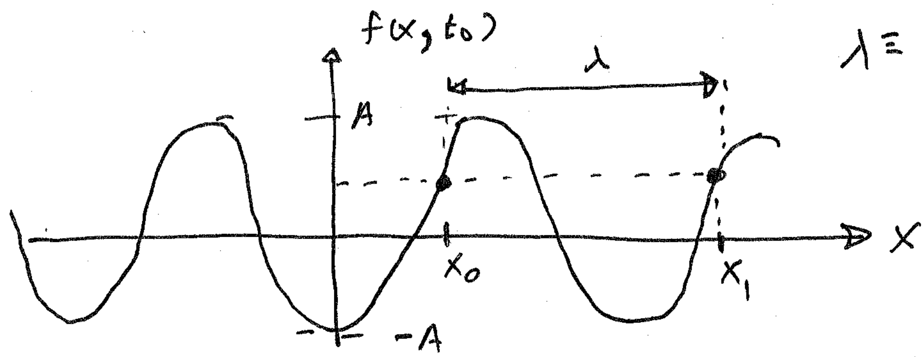
$= A \sin(k(x-vt))$

$A \equiv$ amplitude.

; we haven't ~~specified~~ specified what k must be.

To determine physical meaning of k, consider the following graphs:

① At a fixed time $t = t_0$:



$\lambda \equiv$ wave length

so, $g(z_0) = A \sin(kz_0)$

$g(z_1) = A \sin(kz_1)$

$\left(\begin{array}{l} z_0 \equiv x_0 - vt_0 \\ z_1 \equiv x_1 - vt_0 \end{array} \right)$

And we have from the graph:

$kz_1 = kz_0 + 2\pi$

$\Rightarrow k(z_1 - z_0) = 2\pi$

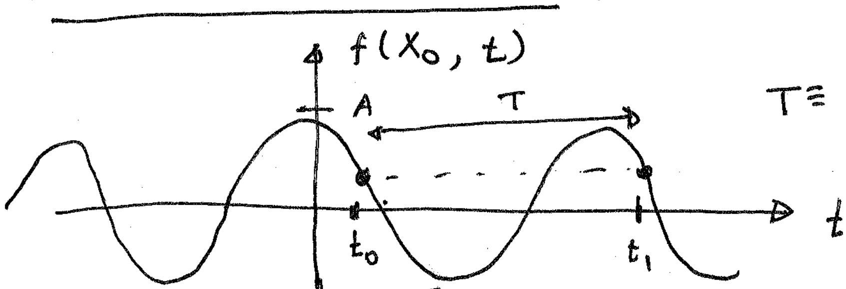
$\Rightarrow k = \frac{2\pi}{(x_1 - vt_0) - (x_0 - vt_0)}$

$= \frac{2\pi}{\lambda}$

Hence, we have just shown $\boxed{k = \frac{2\pi}{\lambda}}$

↑ called "wave number".

② At a fixed location $x = x_0$, stand there and watch the wave pass by over time:



$T \equiv$ period

$g(z_0) = A \sin(kz_0)$

$g(z_1) = A \sin(kz_1)$

From graph:

$kz_1 = kz_0 - 2\pi$

$\Rightarrow k(x_0 - vt_1) - k(x_0 - vt_0) = -2\pi$

$\left(\begin{array}{l} z_0 = x_0 - vt_0 \\ z_1 = x_0 - vt_1 \end{array} \right)$

over →

Continued . . .

$$\Rightarrow \quad -kV\underbrace{(t_1 - t_0)}_{= T} = -2\pi$$ ← Could be $+2\pi$ but I picked -2π to make the final answer be $(+)$. (i.e. want T to be $(+)$)

$$\Rightarrow \quad kV = \frac{2\pi}{T}$$

but $\frac{1}{T} = f$ ← frequency.

$$\Rightarrow \quad kV = 2\pi f = \omega$$

↑ Angular frequency.

So, we've just physically reasoned through and found out that

$$\boxed{kV = \omega} \quad \Rightarrow \quad \boxed{\frac{2\pi}{\lambda} V = \omega}$$

Sinusoidal wave:

$$\therefore \quad f(x,t) = A \sin\left(k(x - vt)\right)$$

$$= A \sin(kx - kvt)$$

$$= \boxed{A \sin(kx - \omega t)}$$ ← sinusoidal wave moving to right. $\longrightarrow v$.

For sinusoidal wave moving to left, we have:

$$f(x,t) = A \sin\left(k(x + vt)\right)$$

$$= A \sin(kx + \omega t)$$ ← This is fine but we often write it (in textbooks) as this

$$= \underbrace{-A}_{= B} \sin(-kx - \omega t)$$

$$= \boxed{B \sin(-kx - \omega t)}$$

So:

$$\boxed{\begin{aligned} f_-(x,t) &= A \sin(kx - \omega t) \quad \longrightarrow v \\ f_+(x,t) &= B \sin(-kx - \omega t) \quad \longleftarrow v \end{aligned}}$$

[Sinusoidal waves]