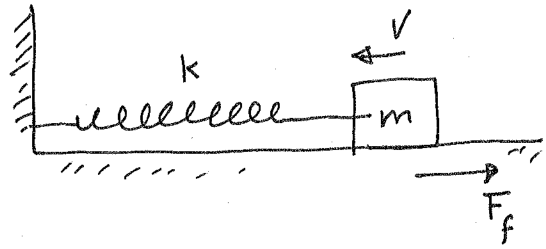
B.) Damped Harmonic Oscillator (i.e. SHM with friction (or some dissipative force.))

EX :



F_f = friction between block and the floor.

- You're probably used to friction being $F_f = \mu mg$ where μ is coefficient of (kinetic / static) friction in high school.

But actually, friction is dependent on velocity of an object.

(e.g. air resistance on an object moving w/ velocity V is $F_f = -bV$.)

$\uparrow$ $\ominus$ because in the direction opposite direction of motion of object.)

So, in this course, we'll use the more realistic version of drag force (friction) :

$$\boxed{F_f = -bV = -b\frac{dx}{dt}. \quad (\text{where } b>0)}$$

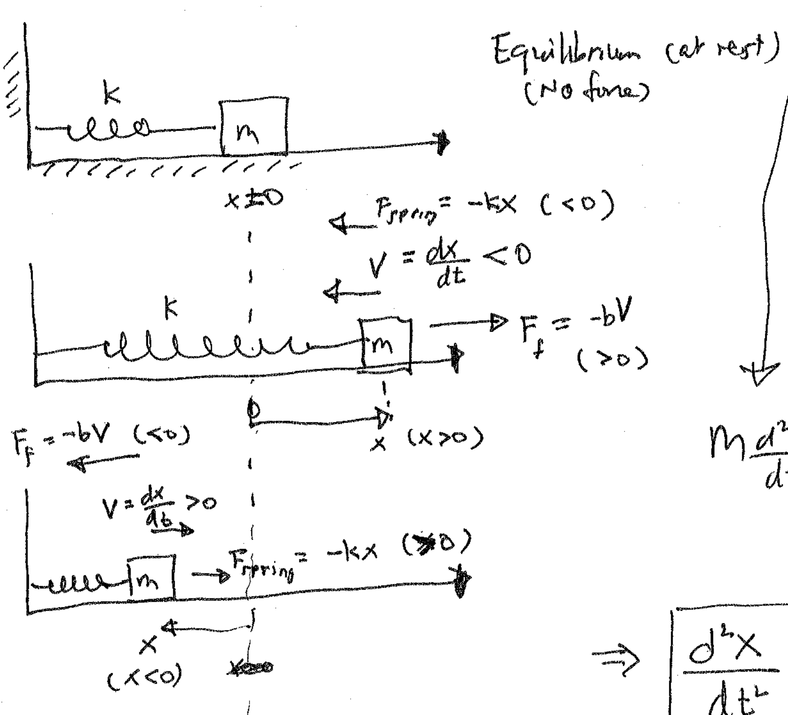
$[b] = \frac{\text{Newton}}{m/s} = \frac{kg \cdot m/s^2}{m/s} = \frac{kg \cdot m}{s}$

$\uparrow$ "dimension" of b.

Note : "b" is called the damping coefficient

(Large "b" means large drag force.)

Thus, Newton's 2nd law on the block now reads :



$$m\frac{d^2x}{dt^2} = F_{net} = F_f + F_{spring}$$
$$= -bV - kx$$
$$= -b\frac{dx}{dt} - kx$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0}$$

$\uparrow$ EOM.

- Just as it was advantageous to consider the general (i.e. somewhat more abstract) form of EOM when we studied SHM w/o friction (damping), we want to consider
$\left(\frac{d^2x}{dt^2} + \omega^2 x = 0\right)$
a general form of EOM which looks like the boxed eq'n (EOM) we derived at the bottom of pg28.

i.e. $$\frac{d^2x}{dt^2} + \underbrace{\left(\frac{b}{m}\right)}_{\substack{\text{call this} \\ = 2\gamma \\ (\gamma \equiv \frac{b}{2m})}} \frac{dx}{dt} + \underbrace{\left(\frac{k}{m}\right)}_{\text{call this } \omega_0^2} x = 0.$$

$\boxed{\omega_0 \equiv \text{"Natural angular frequency"}}$

("Natural" meaning the frequency the oscillator would have if there was no friction.)

Then consider the general EOM: $$\boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0} \leftarrow \boxed{\text{Eq'n } (*)}$$

To solve Eq'n (*) (i.e. To find $x(t)$): we again guess the sol'n. But remember from pg27, we now follow the procedure we outlined there.

So first ①: $\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$ ← original EOM we want to solve. ($x(t) \in \mathbb{R}$)

Next, ②: Write down the $\mathbb{C}$-equivalent version of EOM:

$$\boxed{\text{Eq'n (1)}} \longrightarrow \boxed{\frac{d^2 z}{dt^2} + 2\gamma \frac{dz}{dt} + \omega_0^2 z = 0} \quad (z(t) \in \mathbb{C}).$$

Next, ③: Want to solve for $z(t)$. We guess the sol'n,

we pick $z(t) = Ae^{i\omega t}$ (as we did when we solved the SHM EOM previously).

"A" is an arbitrary constant to be determined by initial condition of system.
"ω" is to be determined.

To check if our guess $z(t) = Ae^{i\omega t}$ can satisfy Eq'n (1), we plug in and see:

over →

so, plugging in $Z(t) = Ae^{i\omega t}$ we get:

$$\frac{d^2 Z}{dt^2} + 2\gamma \frac{dZ}{dt} + \omega_0^2 Z = -A\omega^2 e^{i\omega t} + 2\gamma i\omega \underbrace{Ae^{i\omega t}}_{Z(t)} + \omega_0^2 \underbrace{Ae^{i\omega t}}_{Z(t)}$$

$$= \left[-\omega^2 + 2\gamma i\omega + \omega_0^2\right] Z(t)$$

Now, for $Z(t)$ to be the solution to our $\mathbb{C}$-equivalent EOM, we need

$\left[-\omega^2 + 2\gamma i\omega + \omega_0^2\right] Z(t) = 0.$ for all t.

This can happen if and only if $\boxed{-\omega^2 + 2\gamma i\omega + \omega_0^2 = 0}$ ← $\boxed{\text{Eq'n (2)}}$

Notice that we haven't specified what "ω" should be yet.

(*) The idea here is that by selecting the just right value for "ω", (i.e. one that satisfies Eq'n (2)), $Z(t) = Ae^{i\omega t}$ would indeed be a solution to Eq'n (1) (on pg 29).

so, find what ω should be from eq'n (2): $\omega^2 - 2i\gamma\omega - \omega_0^2 = 0$

↑ quadratic eq'n in ω.

Find the root(s) ω: (recall $a_1 y^2 + a_2 y + a_3 = 0$

$\Rightarrow y_{\pm} = \dfrac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$)

so:

$$\omega_{\pm} = \frac{2i\gamma \pm \sqrt{(-2i\gamma)^2 + 4\omega_0^2}}{2}$$

$$= \frac{2i\gamma \pm \sqrt{4(\omega_0^2 - \gamma^2)}}{2}$$

$$\Rightarrow \boxed{\omega_{\pm} = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}}$$ ← 2 roots (2 possible values of ω)

So which value (ω_+ or ω_-) shall we pick?

Well, consider $Z_+(t) \underset{\uparrow}{\equiv} A_1 e^{i\omega_+ t}$ and $Z_-(t) \equiv A_2 e^{i\omega_- t}$

"define as"

- Notice that both $Z_+(t)$ and $Z_-(t)$ satisfy Eq'n (1) on Pg 29.

  (After all, that's how we arrived at choosing ω_+ and ω_- on Pg 30.)

- Furthermore, consider $Z(t) \equiv Z_+(t) + Z_-(t)$

then:

$$\frac{d^2 Z}{dt^2} + 2\gamma \frac{dZ}{dt} + \omega_0^2 Z$$

$$= \frac{d^2}{dt^2}[Z_+ + Z_-] + 2\gamma \frac{d}{dt}[Z_+ + Z_-] + \omega_0^2 [Z_+ + Z_-]$$

$$= \frac{d^2}{dt^2} Z_+ + \frac{d^2 Z_-}{dt^2} + 2\gamma \frac{dZ_+}{dt} + 2\gamma \frac{dZ_-}{dt} + \omega_0^2 Z_+ + \omega_0^2 Z_-$$

$$= \Big\{ \underbrace{\frac{d^2 Z_+}{dt^2} + 2\gamma \frac{dZ_+}{dt} + \omega_0^2 Z_+}_{=0} \Big\} + \Big\{ \underbrace{\frac{d^2 Z_-}{dt^2} + 2\gamma \frac{dZ_-}{dt} + \omega_0^2 Z_-}_{=0} \Big\}$$

$$= 0$$

Since each: Z_+ and Z_- separately satisfy Eq'n (1).

Thus indeed,

$$\boxed{\begin{aligned} Z(t) &= Z_+(t) + Z_-(t) \\ &= A_1 e^{i\omega_+ t} + A_2 e^{-i\omega_- t} \end{aligned}}$$

is a solution to $\mathbb{C}$-equivalent EOM. (eq'n (1) on Pg 30)

- In fact, it is the most general sol'n to eq'n (1) on Pg 30

  (i.e. Any sol'n to eq'n (1) can be written as $Z(t) = A_1 e^{i\omega_+ t} + A_2 e^{-i\omega_- t}$ by choosing appropriate values of A_1 & A_2.)

- This sol'n $Z(t)$ has 2 degrees of freedom (Just as in SHM w/o friction).

By the same physical reasoning (as applied in SHM w/o friction), these 2 degrees of freedom must correspond to

1.) Amplitude (you decide at $t=0$)

2.) Origin of time (you decide when to start ($t=0$) the timer.)

Now, we're not quite done yet.

$Z(t) = A_1 e^{i\omega_+ t} + A_2 e^{i\omega_- t}$ is the most general sol'n but it's not in the most illuminating form as it's currently written.

To simplify above; notice that (see Pg 30)

$$i\omega_\pm = i\left[i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}\right]$$

$$= -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

And let's define $\tilde{\omega}^2 \equiv \left(\sqrt{\omega_0^2 - \gamma^2}\right)^2$

$$= \omega_0^2 - \gamma^2$$

Thus:

$$i\omega_\pm = -\gamma \pm i\tilde{\omega}$$

Hence,
$$Z(t) = A_1 e^{i\omega_+ t} + A_2 e^{i\omega_- t}$$

$$= A_1 \exp\left[(-\gamma + i\tilde{\omega})t\right] + A_2 \exp\left[(-\gamma - i\tilde{\omega})t\right]$$

$$= \left[\exp(-\gamma t)\right]\left[A_1 e^{i\tilde{\omega} t} + A_2 e^{-i\tilde{\omega} t}\right]$$

Hence:

$$\boxed{Z(t) = \left(\exp(-\gamma t)\right)\left(A e^{i\tilde{\omega} t} + A_2 e^{-i\tilde{\omega} t}\right)}$$

where $\tilde{\omega} \equiv \sqrt{\omega_0^2 - \gamma^2}$

$\gamma \equiv \frac{b}{2m} > 0$

• This form of $Z(t)$ is more illuminating in the following sense:

(*) Consider the magnitude of $\mathbb{C}$-function $Z(t)$, and think about how it's changing as $t \to +\infty$.

So we're interested in $\underbrace{|Z(t)|}_{\text{magnitude of } Z(t).} = \left|\exp(-\gamma t)\right| \cdot \left|A_1 e^{i\tilde{\omega} t} + A_2 e^{-i\tilde{\omega} t}\right|$.

Now, $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$ can be:

3 cases! (depending on strength of damping force)

1. $\tilde{\omega} > 0$: if $\omega_0^2 > \gamma^2$. ("weak" damping)
2. $\tilde{\omega} = 0$: if $\omega_0^2 = \gamma^2$ ("critical" damping).
3. $\tilde{\omega}$ is imaginary : if $\omega_0^2 < \gamma^2$. ("strong" damping)
   (In this case, we write $\tilde{\omega} = i\tilde{\tilde{\omega}}$, $\tilde{\tilde{\omega}} > 0$.)
   (Just as we write $\sqrt{-3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$)

• What does each of the 3 cases mean physically?

Next, we set out to investigate this.

case 1 : "weak" damping (aka : under damped) :

$\tilde{\omega} > 0$: $\Rightarrow$ $\omega_0^2 > \gamma^2$ (recall: $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$

where $\gamma \equiv \frac{b}{2m}$ $b \equiv$ "damping coefficient"

But since $\gamma \propto b$, we can consider γ to be strength of damping force.

In this case, we have: $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$ is $\mathbb{R}$.

and, $Z(t) = e^{-\gamma t}\left(A_1 e^{i\tilde{\omega} t} + A_2 e^{-i\tilde{\omega} t}\right)$

$$= e^{-\gamma t}\left[\underbrace{(A_1 + A_2)}_{\equiv A}\cos(\tilde{\omega} t) + i\,\underbrace{(A_1 - A_2)}_{\equiv B}\sin(\tilde{\omega} t)\right]$$

viz. $e^{i(\cdot)} = \cos(\cdot) + i\sin(\cdot)$

$$= e^{-\gamma t}\left[A\cos(\tilde{\omega} t) + iB\sin(\tilde{\omega} t)\right]$$

~~Otherwise~~

This, we can rewrite as $C\cos(\tilde{\omega} t - \phi)$ as we've seen before in class.

$$\Rightarrow \boxed{x(t) = Ce^{-\gamma t}\cos(\tilde{\omega} t - \phi)}$$

(under damped sol'n)

(Note: went from "$Z(t)$" to "$x(t)$")
$\mathbb{C}$ $\mathbb{R}$.

since at the last step, we wrote down the $\mathbb{R}$ analogue of $Z(t)$)

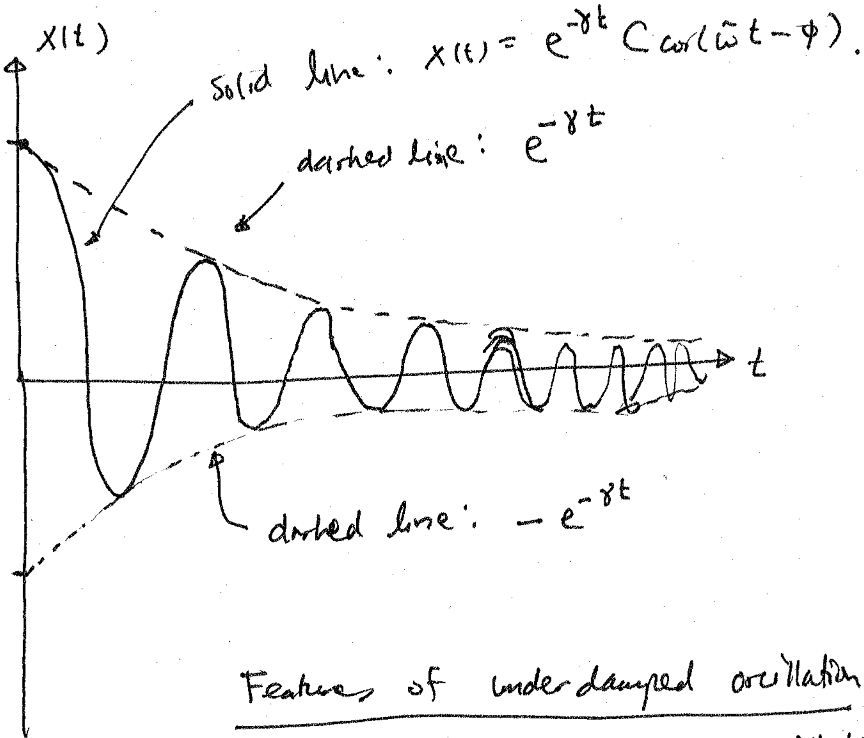
$x(t) = \underbrace{e^{-\gamma t}} x_{SHM}(t)$, where $x_{SHM}(t) = C\cos(\tilde{\omega} t - \phi)$

sol'n to SHM: $\boxed{\ddot{x}_{SHM} + \tilde{\omega}^2 x_{SHM} = 0.}$

exponential decay with characteristic decay time of $\frac{1}{\gamma}$ (Note: $\left[\frac{1}{\gamma}\right]$ = time.)

Measure of ~~amt~~ of time needed for the amplitude of oscillation (Initially "C") to decay by factor of $e^{-1} \simeq 0.37$

So:



Features of underdamped oscillation:

(1) ~~Amplitu~~ particle still oscillates, but now with ever decreasing amplitude ($e^{-\gamma t}$ decay of amplitude).

(2) Angular frequency of oscillation is

$$\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2} < \omega_0.$$

(Hence, friction decreases frequency of oscillation in underdamped situation.)

(3) Total energy: Recall from SHM: $E_{tot} = \frac{1}{2} k A^2$ A = Amplitude.

~~Here~~ But in underdamped case, total energy is decreasing due to friction; and more specifically, amplitude is decreasing: $A(t) = A e^{-\gamma t}$.

Thus:

$$E_{tot} = \frac{1}{2} k \left(A e^{-\gamma t}\right)^2 = \underbrace{\left(\frac{k}{2} A^2\right)}_{\text{energy of simple harmonic motion.}} e^{-2\gamma t} = E_{SHM} e^{-2\gamma t}$$

case 2: Critical damping : $\tilde{\omega} = 0$ (i.e. $\omega_0 = \gamma$.)

In this case, the damping constant "b" is such that $\gamma = \frac{b}{2m}$ matches $\omega_0 = \frac{k}{m}$.

i.e. $b = 2k$

Notice that in this case, $Z(t)$ from pg 33 becomes: $Z(t) = e^{-\gamma t}\underbrace{(A_1 + A_2)}_{C}$
$= Ce^{-\gamma t}$

Alarm! $Z(t)$ has only one free parameter : "C".

But as we learned in class, a 2^{nd} order differential equation

such as our EOM for damped motion : (i.e. Differential equation with highest degree of derivative = 2.)

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

2^{nd} order

So, this means that we haven't found all the solutions to our EOM.

We need one more coefficient. : so far, we found $Ce^{-\gamma t} \equiv Z_1(t)$

In this case, it turns out that $t\,Z_1(t) \equiv Z_2(t)$ is a solution of our EOM :

check : $\ddot{Z}_2 + 2\gamma\dot{Z}_2 + \omega_0^2 Z_2$

$\frac{dZ_2}{dt} = Z_1 + t\dot{Z}_1$

$\frac{d^2Z_2}{dt^2} = \dot{Z}_1 + \dot{Z}_1 + t\ddot{Z}_1$

substitute in

$= 2\dot{Z}_1 + t\ddot{Z}_1 + 2\gamma(Z_1 + t\dot{Z}_1) + \omega_0^2 t Z_1$

$= t\underbrace{(\ddot{Z}_1 + 2\gamma\dot{Z}_1 + \omega_0^2 Z_1)}_{0} + 2\dot{Z}_1 + 2\gamma Z_1$ (since Z_1 satisfies damped EOM).

$= 2[\dot{Z}_1 + \gamma Z_1]$: $\dot{Z}_1 = -\gamma Ce^{-\gamma t}$

$= 2[-\gamma Ce^{-\gamma t} + \gamma Ce^{-\gamma t}]$

$= 0.$

$\therefore$ Indeed, $Z_2(t) = t Z_1(t)$ satisfies damped EOM when $\tilde{\omega} = 0$

and the most general sol'n is: $Z(t) \equiv Z_1(t) + Z_2(t)$

$$= C_1 e^{-\gamma t} + t C_2 e^{-\gamma t}$$
$$= e^{-\gamma t}\left[C_1 + t C_2\right].$$

already $\mathbb{R}$.

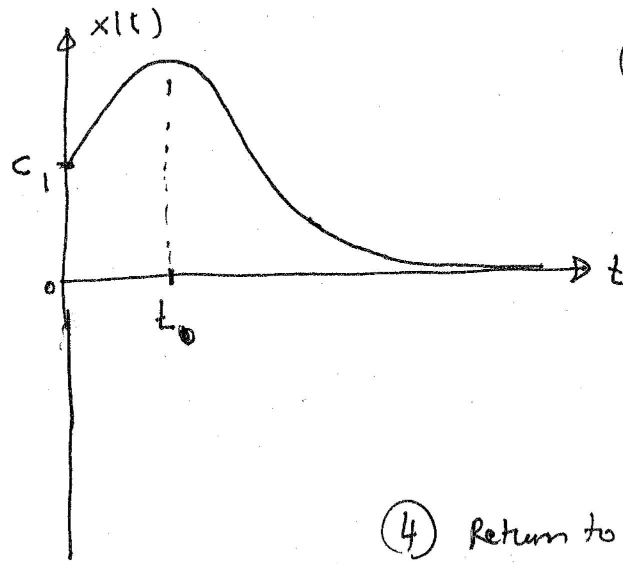
So, instead of writing "$Z(t)$", write

$$\boxed{x(t) = e^{-\gamma t}\left[C_1 + t C_2\right]}$$

Critically damped. sol'n.

(C_1 & C_2: free parameters

Features of critically damped sol'n:



1) No oscillation (contrast w/ underdamped oscillation

2) Initially shoots "up" then ~~exponenti~~ decays towards $x=0$, exponentially.

3) You can find t_0, where $x(t_0)$ is max. of $x(t)$; by solving for $\left.\frac{dx}{dt}\right|_{t=t_0} = 0$.

4) Return to equilibrium is fastest for critical damping (faster than in underdamped & overdamped cases.)
(you'll be asked to show this in Pset 2.)

Note: By "fastest" return to equilibrium $x=0$, we mean w/o oscillating (as in underdamped case) and settles down at $x=0$.

Case 3: ~~Critical damping~~
Overdamped case : $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$ is imaginary #.
$(\omega_0 < \gamma)$.

In this case, call $\tilde{\omega} = i\tilde{\tilde{\omega}}$ where $\tilde{\tilde{\omega}} = \sqrt{\gamma^2 - \omega_0^2} \in \mathbb{R}$.
> 0.

In this case, $z(t)$ of (pg33) becomes:

$$z(t) = e^{-\gamma t}\left[A_1 e^{i(i\tilde{\tilde{\omega}})t} + A_2 e^{-i(\tilde{\tilde{\omega}} i)t}\right]$$

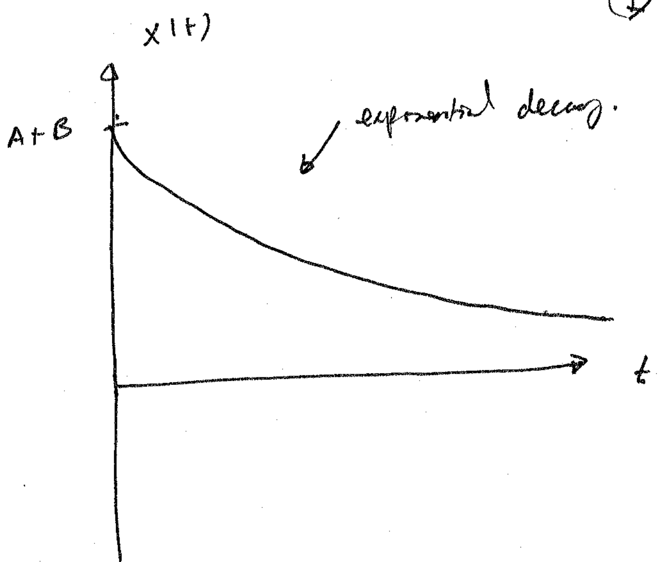
$$= e^{-\gamma t}\left[\underbrace{A_1 e^{-\tilde{\tilde{\omega}} t} + A_2 e^{\tilde{\tilde{\omega}} t}}_{\text{both real}}\right]$$

∴ Instead of writing "$z(t)$", write $x(t)$:

$$\boxed{x(t) = A\exp[-(\gamma + \tilde{\tilde{\omega}})t] + B\exp[-(\gamma - \tilde{\tilde{\omega}})t]}$$

(where, A & B are 2 free parameters for you to choose.)

Features:



① Notice that $\gamma > \tilde{\tilde{\omega}}$
(follows from $\omega_0 < \gamma$)
Hence, both $\gamma + \tilde{\tilde{\omega}}$ and $\gamma - \tilde{\tilde{\omega}}$ are ⊕.
Hence, both $\exp[-(\gamma + \tilde{\tilde{\omega}})t]$ and $\exp[-(\gamma - \tilde{\tilde{\omega}})t]$ are decays
(If $\gamma - \tilde{\tilde{\omega}}$ was ⊖, then we'd get $x(t)$ eventually blowing up as $t \to \infty$.)
physically unreasonable since friction always takes energy away!

② Return (decays) to equilibrium is slower than in critical damping.