MITES 2008 : Physics III - Oscillations and Waves :: Midterm Examination.

Massachusetts Institute of Technology
Instructor: Hyun Youk
Recitation Instructor: Louis Fouche
(Monday, July 7, 2008 : 1:15 - 2:45 PM.)

You have one hour and thirty minutes to complete all the questions in this examination. There are four problems, with each problem containing multiple parts. The last parts of each problem tend to be more challenging than its previous parts. So move on to the next problem if you are stuck on those last parts of a problem.

Each problem is worth **25 points**. Thus this exam is worth **100 points**. But I will grade your exam out of **85 points**. This means that your goal is to get either 85 points or over on this exam by solving as many problems as you can.

Since this is a timed exam, your solutions do not have to be as "organized" as your problem set solutions. You do not have to show any more work than what is necessary for you to get your answers. But be warned: if you're unsure about your solution and you want some partial credit, you do need to have written down some work.

Unless the question asks you to "derive" a solution, you can just state what the solution is (eg. "Write the solution to the EOM" means you can just state the solution without derivation).

The following formulas may be useful:

$$exp(i\alpha t) = cos(\alpha t) + isin(\alpha t)$$

$$\text{Real part of} \quad Aexp(i\omega t) + Bexp(-i\omega t) \quad \text{is} \quad Ccos(\omega t - \phi)$$

$$Ccos(\omega t - \phi) = Ccos(\omega t)cos(\phi) + Csin(\omega t)sin(\phi)$$

$$\text{Quadratic equation:} \quad ax^2 + bx + c = 0 \quad \Rightarrow \quad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$cos^2(\theta) + sin^2(\theta) = 1$$

$$sin(-\theta) = -sin(\theta) \qquad cos(-\theta) = cos(\theta)$$

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Problem 1. Optical tweezer and DNA {Total = 25 points}:

Optical tweezer is a device that traps dielectric microspheres using a finely focused laser beam. By sending a beam with just the right intensity profile, the microsphere (bead) can be trapped within a simple harmonic potential well formed by the laser beam, with the center of the beam being the minimum of the potential well. That is, if the bead is displaced by x from the trap center, a restoring force of $-k_{trap}x$ acts on the bead in an attempt to return it to the trap center ($x = 0$). We can also attach polymers such as DNA to the bead, with one end of the DNA tethered to a stationary glass slide. As long as we are dealing with small stretch and compression of DNA, we can model DNA as a Hookian spring-like object with an "effective" spring constant k_{DNA}. Thus, the system shown in Fig. 1(b) can be modeled as a bead of mass m attached to two springs (with spring constants k_{DNA} and k_{trap}) as shown in Fig. 1 (c) (Figure to the extreme right).

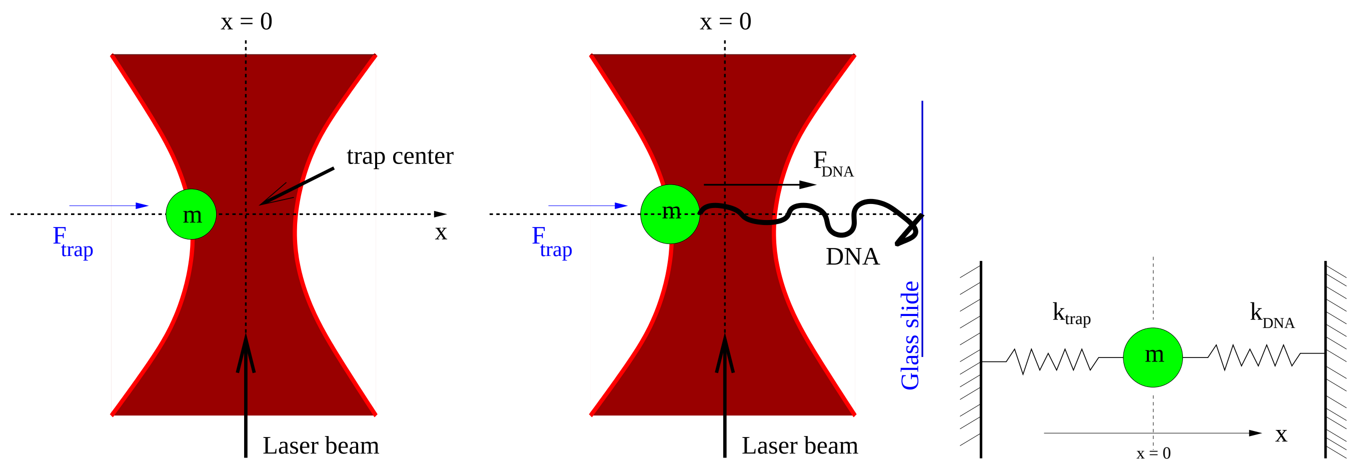

FIG. 1: (a) Left. (b) Middle. (c) Right.: (a). Optical tweezer exerts a restoring force ($-k_{trap}x$) on the microsphere (bead). (b) One end of DNA is attached to the bead while its other end is tethered to a stationary glass slide. DNA also exerts a restoring force ($-k_{DNA}x$) on the bead. (c). The bead in figure1 (b) acts like a block of mass m attached to two springs, with spring constants k_{trap} and k_{DNA}.

(a.) {**5 points**}: Derive the equation of motion for the bead (shown in Fig.1 (c)) and thus show that the bead acts as a simple harmonic oscillator. State what the *natural angular frequency* ω_0 is, in terms of k_{DNA}, k_{trap} and m. You can assume that $x = 0$ is the equilibrium position of the bead and that both springs are at their rest lengths at $x = 0$.

(b.) {**2 points**}: Write down the general solution for the equation of motion you derived in (a.). You do not have to derive the solution; you can just state it. Define the physical meaning of all the free parameters in your solution.

(c.) {**6 points**}: Write down the total energy of the system shown in Fig.1(c). At what value(s) of x is the kinetic energy maximum? At what value(s) of x is the potential energy of "DNA spring" maximum? At what value(s) of x is the potential energy of "trap" spring maximum? Answer in terms of one of the free parameters you defined in (b). Also, what is the ratio of potential energy stored in the "trap - spring" to the potential energy stored in the "DNA spring" at time t?

(d.) {**5 points**}: Derive the equation of motion again, but this time, use conservation of the total energy you found in (c.).

(e.) {**4 points**}: Suppose that at $t = 0$, the bead is at $x(t = 0) = 0$ and that its instantaneous velocity is V_0. By finding specific values for the free parameters in your solution $x(t)$ found in (b), write down $x(t)$ that describes what the position of the bead is for subsequent times (t > 0).

(f.) {**3 points**}: If the DNA suddenly "breaks" (i.e. spring with stiffness k_{DNA} is suddenly cut with scissors while the bead is oscillating) at the instant the bead is at $x = 0$ (equilibrium position), how much energy is lost from the system?

Problem 2. Heating up a pole using gravity. {**Total = 25 points**}:

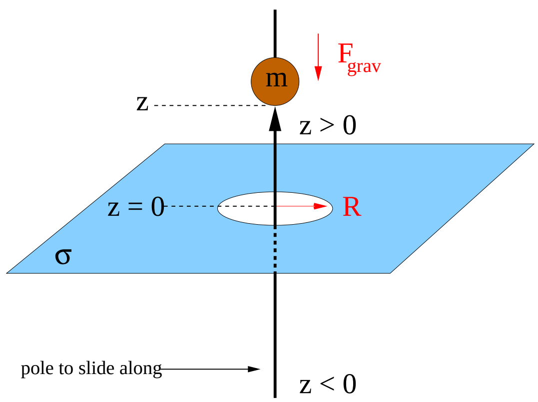

FIG. 2: Bead of mass m slides along the vertical bar (z-axis). The bar provides frictional force $F_{drag} = -b\dot{z}$. The force of gravity due to the infinitely large sheet of uniform mass density σ is $F_{grav} = -\dfrac{2\pi G\sigma mz}{R}$. Notice that $z(t)$ is the position of particle. z(t) > 0 if particle is above the sheet of mass, and z(t) < 0 if its below. $z = 0$ is the center of the hole in the plane of sheet of mass.

An infinite sheet of paper has a uniform mass density (per area) σ. A circular hole of radius R is cut through the paper, allowing a small particle of mass m to go through it. It turns out, just like the tunnel through Earth on Problem set #1, the particle oscillates up and down vertically through the hole, as long as its displacement along the z-axis is not too large. This is shown in Fig. 2. If the particle is at position z, then the sheet of paper exerts a *restoring force* (gravitational force) of

$$F_{grav} = -\frac{2\pi G\sigma mz}{R} \tag{1}$$

in an attempt to bring the particle back to $z = 0$ (equilibrium position). Suppose that the particle is sliding on the vertical pole (z-axis) which causes *damping* on the particle by $F_{damping} = -b\dfrac{dz}{dt}$, where b is a *damping constant.*

(a.) {**3 points**}: Starting from Newton's second law, show that the equation of motion for the particle has the form

$$\frac{d^2z}{dt^2} + 2\gamma\frac{dz}{dt} + \omega_0^2 z = 0 \tag{2}$$

Identify what the constants γ and ω_0 are, in terms of the parameters given in the problem. Under what condition(s) does the particle oscillate with angular freuqnecy ω_0?

(b.) {**5 points**}: The complex-equivalent equation of motion for this system is

$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + \omega_0^2 Q = 0 \tag{3}$$

By guessing the solution to be $Q(t) = Aexp(i\omega t)$, derive a relationship ω has to satisfy in order for our guess $Q(t)$ to be indeed a solution to above equation of motion.

(c.) {**10 points**}: Based on your solution to (b.), state the conditions for the following regimes of motion in terms of ω_0 and γ. Also, provide a sketch of position (z(t) = Real($Q(t)$)) vs. time for each of the three cases.
(i,) "Weak" damping (i.e. Underdamped motion)
(ii.) Critical damping (i.e. critically damped motion)
(iii.) "Strong" damping (i.e. overdamped motion)

(d.) {**4 points**}: Show that critically damped motion indeed approaches equilibrium faster than the other two cases. To do this, look at the exponential decay of amplitude associated with each of the three cases. You just have to comment which one decays faster than others. No calculus (derivatives and such) needed here.

(e.) {**3 points**}: Suppose the particle is underdamped. In this case, the position of the particle along the pole is described by $z(t) = Cexp(-\gamma t)cos(\omega t - \phi)$. What is the maximum amount of heat energy that the pole can gain from the sliding particle after the particle has oscillated for time interval $t = \dfrac{1}{2\gamma}$? Here, we're assuming that part of energy lost due to damping is used to increase the thermal energy of the pole.

Problem 3. Simple harmonic oscillator with sinusoidal external force. {Total = 25 points}:

In class, we looked at an equation of motion describing a system with a damping force, restoring force, and an external sinusoidal driving force $F(t) = F_0 cos(\omega t)$, where ω is the angular frequency of the driving force $F(t)$. From class (and in your lecture notes), we found that the equation of motion describing the motion of such a particle is

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}cos(\omega t), \tag{4}$$

where $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the natural angular frequency associated with the spring (spring constant k) and the particle of mass m, and $\gamma = \dfrac{b}{2m}$ represents the strength of damping (b is the damping constant).

(a,) {**3 points**}: Write down the complex-equivalent equation of motion corresponding to equation (4).

(b.) {**6 points**}: Let's guess a solution to the complex-equivalent equation of motion (EOM) to be $z_p(t) = Aexp(i\omega t)$. Show that this is indeed a solution of the EOM provided that A obeys a relationship you derive in the process.

(c.) {**3 points**}: $z_p(t)$ is only a *particular* solution to the equation of motion. State what we mean by the *linearity* (aka. *superposition principle*) property of the EOM (*Hint*: What does it mean for a differential equation to be *linear*)? Using this linearity principle, write down the *general solution* to the complex-equivalent equation of motion.

(d.) {**10 points**}: When the system is *underdamped*, the *real* part of the general solution you found in (c.) becomes

$$x(t) = Re(z(t)) = Ce^{-\gamma t}cos(\tilde{\omega}t - \phi) + \frac{F_0}{m}\frac{(\Omega^2 cos(\omega t) + 2\gamma\omega sin(\omega t))}{(\Omega^4 + 4\gamma^2\omega^2)}, \tag{5}$$

where $\Omega \equiv \sqrt{\omega_0^2 - \omega^2}$. Taking above equation (5) for granted, calculate the *resonant driving angular frequency* $\omega = \omega_r$ from above expression. That is, what is the driving frequency ω that maximizes the amplitude of the following:

$$x_p(t) = \frac{F_0}{m}\frac{(\Omega^2 cos(\omega t) + 2\gamma\omega sin(\omega t))}{(\Omega^4 + 4\gamma^2\omega^2)} \tag{6}$$

To do this, write above equation as $x_p(t) = D(\omega)cos(\omega t - \theta)$, where $D(\omega)$ is the amplitude of $x_p(t)$ that's a function of ω you need to determine. θ is a free parameter. Then find ω that maximizes $D(\omega)$. That ω is the resonant frequency ω_r.

(e.) {**3 points**}: In the underdamped case, the displacement of particle is $x(t)$ given in equation (5). After long enough time $t \gg 1/\gamma$, what is the total energy of the particle?

Problem 4. Coupled oscillator. {**Total = 25 points**}:

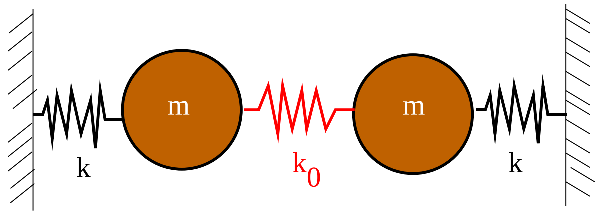

FIG. 3: Coupled oscillator: 2 particles of equal mass m interact with each other through the middle spring whose spring constant is k_0. Each of the two particles are also each joined to a wall through a spring with spring constant k.

(a.) {**4 points**}: Derive the equations of motion for the system in Fig. 3. Let $x_1(t)$ and $x_2(t)$ denote the position of left and right particle at time t respectively. And assume that $x_1 = 0$ and $x_2 = 0$ are the equilibrium positions of left and right particles respectively.

(b.) {**2 points**}: Write the equations of motion in a matrix form. The final answer should be in following form:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7}$$

where you need to determine what a, b, c, and d are.

(c.) {**5 points**}: By writing down the complex-number equivalent equation of motion, then guessing the solution to be

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} exp(i\alpha t) \tag{8}$$

show that the complex-number equivalent equation of motion becomes

$$\begin{pmatrix} -\alpha^2 + \omega_0^2 + \omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + \omega_0^2 + \omega^2 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

where $\omega_0^2 = \dfrac{k}{m}$ and $\omega^2 = \dfrac{k_0}{m}$.

(d.) {**4 points**}: Recall from class that the "interesting" (non-trivial) solution to EOM (eqn (9)) occurs if and only if the determinant of matrix P is zero. By setting $Det(P) = 0$, find what value(s) α should be, in order for that to happen. The following formula will be useful:

$$Det(P) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = p_{11}p_{22} - p_{12}p_{21} \tag{10}$$

(e.) {**4 points**}: In (d), you should have found two different values of α (four, if you count the sign differences). Using these *normal mode angular frequencies* α, it turns out that the real, physically meaningful general solution is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} cos(\omega_0 t - \phi_1) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} cos(\omega_1 t - \phi_2), \tag{11}$$

where $\omega_0 \equiv \sqrt{\dfrac{k}{m}}$, and $\omega_1 \equiv \sqrt{\dfrac{k + 2k_0}{m}}$. A_1, A_2, ϕ_1, and ϕ_2 are the four free parameters describing the two particles. What kind of motion does each one of these two normal modes (ω_0 oscillation, and the ω_1 oscillation) describe?

(f.) {**3 points**}: Suppose $A_2 = 0$. So the motion of the two blocks is entirely described by the normal mode with angular frequency ω_0. While the blocks are oscillating, suppose we cut the middle spring suddenly (one with spring constant k_0). What happens to the motion of both blocks? How much energy is lost due to the cutting of the middle spring?

(g.) {**3 points**}: Suppose $A_1 = 0$. So the motion of the two blocks is entirely described by the normal mode with frequency ω_1. Now suppose we cut the middle spring when the left block is instantaneously at rest with the left-most spring compressed by A_2 relative to its rest length (its rest length occurs when $x_1 = 0$ and $x_2 = 0$). How much energy is lost due to the cutting of the middle spring?