MITES 08 : Physics III

Midterm Exam Solutions

Mon, July 7, 08

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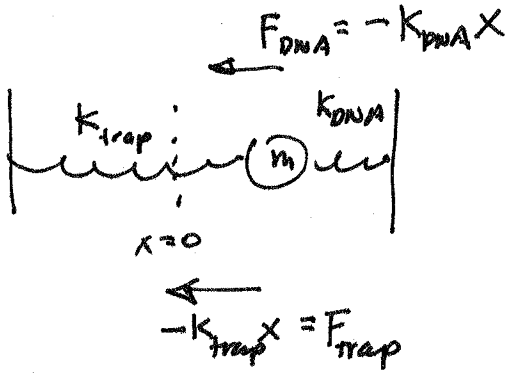
Problem 1.)



1(a)

EOM :

$$m\ddot{x} = F_{DNA} + F_{trap}$$

$$= -k_{DNA}x + (-k_{trap}x)$$

$$\Rightarrow \boxed{\ddot{x} + \underbrace{\frac{(k_{DNA} + k_{trap})}{m}}_{= \omega_0^2} x = 0}$$

So: $\boxed{\omega_0 = \sqrt{\frac{k_{DNA} + k_{trap}}{m}}}$

← Natural angular frequency.

1(b) General sol'n :

$$\boxed{x(t) = C\cos\left(\sqrt{\frac{k_{DNA} + k_{trap}}{m}}\, t - \phi\right)}$$

$C \equiv$ Amplitude

$\phi \equiv$ phase shift

1(c) Total energy :

$$\boxed{E_{tot} = \frac{k_{DNA}x^2}{2} + \frac{k_{trap}x^2}{2} + \frac{m\dot{x}^2}{2}}$$

~~When~~ Since E_{tot} is constant :

$$KE = \frac{m\dot{x}^2}{2} = E_{tot} - \underbrace{\left(\frac{k_{DNA} + k_{trap}}{2}\right)x^2}$$

↑ max if this is minimized.

This occurs only when $x = 0$.

$\Rightarrow$ $\boxed{\text{KE maximum when } x=0}$

- PE of "DNA spring" maximum when: $\boxed{x = \pm C}$

  ← ± Amplitude positions

- PE of "trap spring" maximum when: $\boxed{x = \pm C}$

- $\dfrac{\text{PE of trap}}{\text{PE of DNA}} = \dfrac{k_{trap}x^2}{k_{DNA}x^2} = \boxed{\dfrac{k_{trap}}{k_{DNA}}}$ (at all time t) (ignoring $x=0$ case.)

1(d) * See bottom corner for (d)

(e)

$0 = x(t=0) = C\cos(\phi)$ ... ①

$\dot{x}(t=0) = V_0 = -\omega_0 C\sin(\omega_0 t - \phi)$

$= \omega_0 C\sin(\phi)$ ... ②

So Eq'n ① says: $0 = C\cos(\phi)$ [But $C \neq 0$; want non-trivial sol'n)

$\Rightarrow \cos\phi = 0$

$\Rightarrow \phi = \pi/2$, or $3\pi/2$, or ... etc.

$\Rightarrow$ Eq'n ② becomes:

$V_0 = \omega_0 C\sin(\pi/2)$ (Let's pick $\phi = \pi/2$)

$\Rightarrow C = V_0/\omega_0$

$\therefore x(t) = \dfrac{V_0}{\omega_0}\cos(\omega_0 t - \pi/2)$

$= \boxed{\dfrac{V_0}{\omega_0}\sin(\omega_0 t)}$

(d) $\dfrac{dE_{tot}}{dt} = 0$

$\Rightarrow 0 = \dfrac{d}{dt}\left(\dfrac{m\dot{x}^2}{2} + \dfrac{K_{tot}x^2}{2}\right)$

$\Rightarrow 0 = m\dot{x}\ddot{x} + \cancel{2}K_{tot}x\dot{x}/\cancel{2}$

$\Rightarrow 0 = \dot{x}\{m\ddot{x} + \cancel{K_{tot}}x\}$

But $\dot{x}(t) \neq 0$ for some t. so need:

$\boxed{0 = m\ddot{x} + (K_{DNA} + K_{trap})x}$

PS3

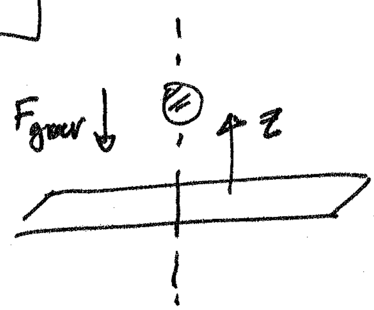
1(f) When $x=0$, no PE stored in DNA since

$$PE_{DNA} = \frac{K_{DNA} x^2}{2}\bigg|_{x=0} = 0.$$

Hence, no energy is lost from the system when DNA breaks.

Problem 2

1(a)



EoM:

$$m\ddot{z} = -\frac{2\pi G \sigma m z}{R} - b\dot{z}$$

$$\Rightarrow \boxed{\ddot{z} + \frac{b}{m}\dot{z} + \left(\frac{2\pi G \sigma}{R}\right) z = 0}$$

← EoM.

So it has form $\boxed{\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = 0}$

where $\boxed{\gamma = \frac{b}{2m}}$, $\boxed{\omega_0 = \sqrt{\frac{2\pi G \sigma}{R}}}$

The particle never oscillates w/ angular freq. ω_0 due to damping.

1(b) $Q(t) = Ae^{i\omega t}$

Then: $\ddot{Q} + 2\gamma \dot{Q} + \omega_0^2 Q$

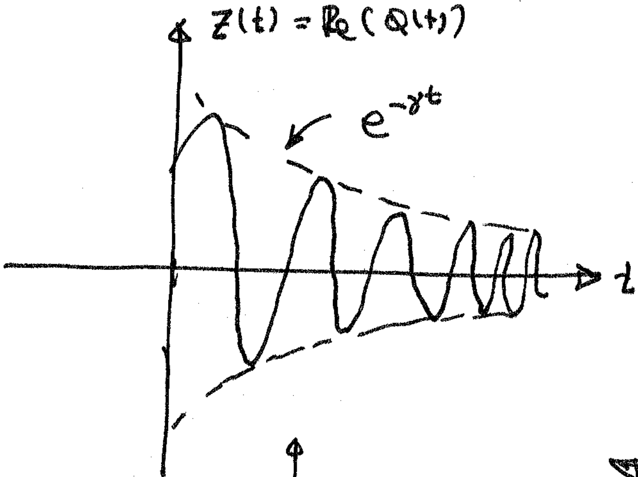
$= -\omega^2 Q + 2\gamma i \omega Q + \omega_0^2 Q \overset{\text{want.}}{=} 0$

$\Rightarrow$

$$-\omega^2 + 2\gamma i \omega + \omega_0^2 = 0$$

$$\Rightarrow \omega = \frac{-2\gamma i \pm \sqrt{-4\gamma^2 + 4\omega_0^2}}{-2} = \boxed{i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}} = \omega_\pm.$$

(c) (i) Weak damping : (underdamped) : $\boxed{\omega_0 > \gamma}$



$$Q(t) = e^{-\gamma t}\left[A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t}\right]$$

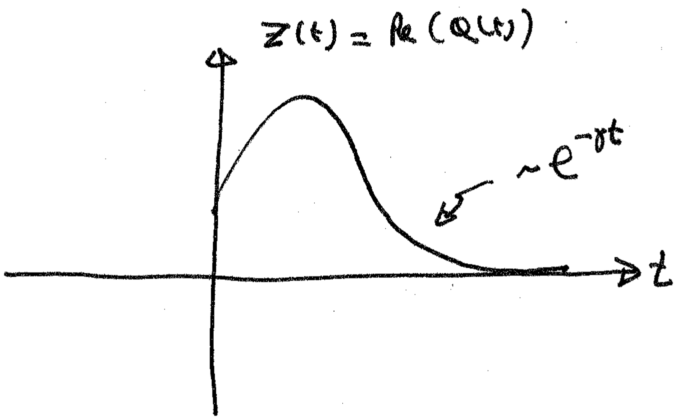
where $\tilde{\omega} \equiv \sqrt{\omega_0^2 - \gamma^2}$

so:

$$z(t) = \mathrm{Re}\,(Q(t)) = e^{-\gamma t} C \cos(\tilde{\omega} t - \phi).$$

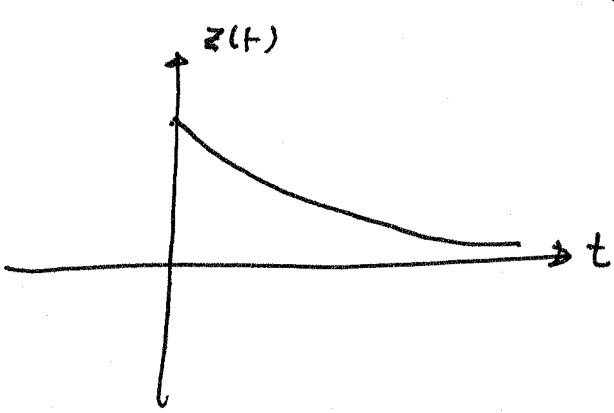
oscillate w/ ang. frequency $\tilde{\omega}$.

(ii) Critical damping : $\boxed{\omega_0 = \gamma}$



$$z(t) = e^{-\gamma t}(A + Bt).$$

(iii) Overdamping : $\boxed{\omega_0 < \gamma}$



$$z(t) = e^{-\gamma t}\left[A e^{\tilde{\tilde{\omega}} t} + B e^{-\tilde{\tilde{\omega}} t}\right] = B e^{-(\gamma + \tilde{\tilde{\omega}})t} + A e^{-(\gamma - \tilde{\tilde{\omega}})t}$$

where $\tilde{\tilde{\omega}} = \sqrt{\gamma^2 - \omega_0^2}$

1(d) The solution in each of 3 cases are:

(i) Underdamped : $z(t) = e^{-\gamma t} C \cos(\tilde{\omega} t - \phi)$

(ii) Critical : $z(t) = e^{-\gamma t} [A + Bt]$

(iii) Overdamped : $z(t) = e^{-\gamma t}[A e^{\tilde{\tilde{\omega}} t} + B e^{-\tilde{\tilde{\omega}} t}]$

$= A e^{-(\gamma - \tilde{\tilde{\omega}})t} + B e^{-(\gamma + \tilde{\tilde{\omega}})t}$

* We want $z(t)$ to reach equilibrium ($z=0$) fastest without oscillating. So we can count out the underdamped case.

So, just compare (ii) & (iii).

(ii) : $e^{-\gamma t}$

(iii) : $A e^{-(\gamma - \tilde{\tilde{\omega}})t} + B e^{-(\gamma + \tilde{\tilde{\omega}})t}$

Now, $\gamma - \tilde{\tilde{\omega}}$ is positive

Reason

(since in overdamped case: $\omega_0 < \gamma$

$\tilde{\tilde{\omega}} = \sqrt{-\omega_0^2 + \gamma^2}$

$\Rightarrow \tilde{\tilde{\omega}}^2 = \gamma^2 - \omega_0^2$

$\Rightarrow \omega_0^2 = \gamma^2 - \tilde{\tilde{\omega}}^2$

we know ω_0 is $+$

so we must have

$\omega_0 = \sqrt{\gamma^2 - \tilde{\tilde{\omega}}^2} > 0$ (and real).

$\therefore \gamma - \tilde{\tilde{\omega}} > 0$

So $e^{-(\gamma - \tilde{\tilde{\omega}})t}$ decays over t.

(So does $e^{-(\gamma + \tilde{\tilde{\omega}})t}$)

since $\gamma + \tilde{\tilde{\omega}}$ is positive.

But since $(\gamma + \tilde{\tilde{\omega}}) > \gamma$,

$e^{-(\gamma + \tilde{\tilde{\omega}})t}$ decays faster than $e^{-\gamma t}$.

i.e.

$$\lim_{t \to \infty} \frac{e^{-(\gamma + \tilde{\tilde{\omega}})t}}{e^{-\gamma t}} = \lim_{t \to \infty} e^{-\tilde{\tilde{\omega}} t} = 0.$$

But, $0 < \gamma - \tilde{\tilde{\omega}} < \gamma$ so $e^{-(\gamma - \tilde{\tilde{\omega}})t}$ decays <u>slower</u> than $e^{-\gamma t}$

In the limit of $t \to \infty$.

i.e. $$\lim_{t\to\infty} \frac{e^{-(\gamma - \tilde{\tilde{\omega}})t}}{e^{-\gamma t}} = \lim_{t\to\infty} e^{\tilde{\tilde{\omega}} t} = +\infty .$$

thus, even though $e^{-(\gamma + \tilde{\tilde{\omega}})t}$ decays faster than $e^{-\gamma t}$,

the other term $e^{-(\gamma - \tilde{\tilde{\omega}})t}$ lags behind $e^{-\gamma t}$,

so in the long t $(t \to \infty)$ regime,

(ii) looks like: $z(t) \sim e^{-\gamma t}$

(iii) looks like: $z(t) \sim e^{-(\gamma - \hat{\omega})t}$ ← slower decay

$\therefore$ $\boxed{\text{Critically damped case reaches equilibrium fastest.}}$

$\boxed{\text{(e)}}$ Maximal energy is: (at time t) $E_{max}(t) = \frac{1}{2} K_{eff} \underbrace{C^2 e^{-2\gamma t}}_{(\text{Amplitude at time } t)^2}$

where K_{eff} is the <u>effective spring constant</u>:

i.e. $$\omega_0 = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{2\pi G \sigma}{R}}$$

$$\Rightarrow \quad k_{eff} = \frac{2\pi G \sigma m}{R}$$

$$\Delta E_{max} \stackrel{\therefore}{=} -E_{max}\left(t = \frac{1}{2\gamma}\right) + E_{max}(t=0) = \frac{k_{eff} C^2}{2}\left[-e^{-1} + e^{0}\right]$$

maximum amt. of heat energy pole can absorb. → $= \boxed{\frac{\pi G \sigma m}{R} C^2 \left[1 - e^{-1}\right]}$

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$\boxed{\text{Problem 3}}$

$\boxed{1(a)}$ The $\mathbb{C}$ - equivalent EOM is : $\boxed{\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} \cos(\omega t).}$

where $z(t)$ is a $\mathbb{C}$ - valued function.

$\boxed{1(b)}$ Before solving, represent : $\cos(\omega t) = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$:

$$\Rightarrow \quad \ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = \frac{F_0}{2m} \left\{ e^{i\omega t} + e^{-i\omega t} \right\}.$$

Then, let's Solve : $\ddot{z}_1 + 2\gamma \dot{z}_1 + \omega_0^2 z_1 = \dfrac{F_0}{2m} e^{i\omega t}$

by guessing $z_1(t) = A e^{i\omega t}$ ← Guess

Plugging in Check : $\ddot{z}_1 + 2\gamma \dot{z}_1 + \omega_0^2 z_1$

$$= -\omega^2 A e^{i\omega t} + 2\gamma i\omega e^{i\omega t} A + \omega_0^2 A e^{i\omega t}$$

$$\Rightarrow \quad A\cancel{e^{i\omega t}} \left[-\omega^2 + 2\gamma i\omega + \omega_0^2 \right] = \frac{F_0}{2m} \cancel{e^{i\omega t}}$$

↑ want

$\Rightarrow$

Need : $\boxed{A = \dfrac{F_0}{2m\left[-\omega^2 + 2i\gamma\omega + \omega_0^2 \right]}}$

~~(And for~~

1(c)

<u>Linearity</u>: If z_1 is sol'n to EOM, and so is z_2, then $z_1 + z_2$ is also a solution to EOM.

This is useful to us since:

$z_1(t)$ (found in (b)) is solution to:

$$\ddot{z}_1 + 2\gamma \dot{z}_1 + \omega_0^2 z_1 = \frac{F_0}{2m} e^{i\omega t}$$

and if we have $z_2(t)$ being solution to: (we can get $z_2(t)$ from z_1 by just changing $\omega \to -\omega$)

$$\ddot{z}_2 + 2\gamma \dot{z}_2 + \omega_0^2 z_2 = \frac{F_0}{2m} e^{-i\omega t}$$

then $z_p \equiv z_1 + z_2$ satisfies:

$$\ddot{z}_p + 2\gamma \dot{z}_p + z_p \omega_0^2 = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \}$$

our original $\mathbb{C}$-equivalent eq'n of motion

But $z_p(t)$ <u>has no free parameters</u>.

~~Next~~ But note that:

$$\{ \ddot{z}_{\text{dmm}} + 2\gamma \dot{z}_{\text{dmm}} + \omega_0^2 z_{\text{dmm}} = 0 \}$$

and

$$+ \{ \ddot{z}_p + 2\gamma \dot{z}_p + z_p \omega_0^2 = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \} \}$$

$$\rightarrow \quad \ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \}.$$

where $\boxed{z = z_{\text{dmm}} + z_p}$ $\leftarrow$ $z_{\text{dmm}}(t) = e^{-\gamma t} \{ A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t} \}$

found in <u>Q #2</u>.

So, using linearity again, we get the general soln:

$$\boxed{\begin{aligned} Z(t) = \; & e^{-\gamma t}\left\{ A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t} \right\} \\ & + \frac{F_0 e^{i\omega t}}{2m}\left\{ \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2} \right\} \\ & + \frac{F_0 e^{-i\omega t}}{2m}\left\{ \frac{1}{-\omega^2 - 2i\gamma\omega + \omega_0^2} \right\} \end{aligned}}$$

where $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$. A & B are free parameter.

$\boxed{(d)}$

$$X_p(t) = \frac{F_0}{m} \; \frac{\left\{ \Omega^2 \cos(\omega t) + 2\gamma\omega \sin(\omega t) \right\}}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (1)$$

want to write as $X_p(t) = D(\omega)\cos(\tilde{\omega} t - \theta)$

$$= D(\omega)\cos(\tilde{\omega} t)\cos(\theta) + D(\omega)\sin(\tilde{\omega} t)\sin(\theta)$$

So, matching $\sin(\omega t)$ and $\cos(\omega t)$ terms in eqn (1) with these:

we have:

$$D(\omega)\cos\theta = \frac{F_0}{m}\frac{\Omega^2}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (2)$$

$$D \sin\theta = \frac{F_0}{m}\frac{2\gamma\omega}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (3)$$

$(2)^2 + (3)^2$:

$$D^2 = \left(\frac{F_0}{m}\right)^2 \frac{\Omega^4 + 4\gamma^2\omega^2}{(\Omega^4 + 4\gamma^2\omega^2)^2} = \left(\frac{F_0}{m}\right)^2 \frac{1}{\Omega^4 + 4\gamma^2\omega^2}$$

$$\Rightarrow \boxed{D(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\Omega^4 + 4\gamma^2\omega^2}}}$$

over →

To maximize $D(\omega)$: minimize the denominator

$$\Omega^4 + 4\gamma^2\omega^2 .$$

$$\Omega \equiv \sqrt{\omega_0^2 - \omega^2}$$

$$\Rightarrow \quad 0 = \frac{d(\Omega^4 + 4\gamma^2\omega^2)}{d\omega}$$

$$= \frac{d}{d\omega}\left\{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right\}$$

$$= 2(\omega_0^2-\omega^2)(-2\omega) + 8\gamma^2\omega$$

$$\Rightarrow \quad -4\cancel{\omega}(\omega_0^2-\omega^2) = -8\gamma^2\cancel{\omega}$$

($\omega \neq 0$ since we have oscillating force)

$$\omega_0^2 - \omega^2 = 2\gamma^2$$

$$\Rightarrow \quad \omega^2 = \omega_0^2 - 2\gamma^2$$

$$\Rightarrow \quad \boxed{\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}}$$

← resonant angular frequency

(e) $t \gg \frac{1}{\gamma}$: Eq'n (5) on midterm handout becomes:

$$X(t) \approx X_p(t).$$

so: $$\boxed{E_{total}(t) = \frac{1}{2}kX_p(t)^2 + \frac{m\dot{X}_p^2}{2}}$$

Pg 11

Problem 4

(a)

$$\boxed{\begin{aligned} m\ddot{x}_1 &= -kx_1 + k_0(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - k_0(x_2 - x_1) \end{aligned}}$$ ← EOMs.

(b) Expanding out above EOMs:

$$\begin{aligned} \ddot{x}_1 &= -\frac{k}{m}x_1 - \frac{k_0}{m}x_1 + \frac{k_0}{m}x_2 \\ &= -\omega_0^2 x_1 - \omega^2 x_1 + \omega^2 x_2 \\ &= -(\omega_0^2 + \omega^2)x_1 + \omega^2 x_2 \end{aligned}$$

$$\ddot{x}_2 = +\omega^2 x_1 - (\omega_0^2 + \omega^2)x_2$$

$$\Rightarrow \boxed{\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} -(\omega_0^2 + \omega^2) & \omega^2 \\ \omega^2 & -(\omega_0^2 + \omega^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

(c) $\mathbb{C}$ - equivalent EOMs in matrix form:

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = \begin{bmatrix} -(\omega_0^2 + \omega^2) & \omega^2 \\ \omega^2 & -(\omega_0^2 + \omega^2) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Guess :

$$\begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} e^{i\alpha t}$$

Check :

$$-\alpha^2 \begin{bmatrix} A \\ B \end{bmatrix} \cancel{e^{i\alpha t}} = \begin{bmatrix} -(\omega_0^2+\omega^2) & \omega^2 \\ \omega^2 & -(\omega_0^2+\omega^2) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \cancel{e^{i\alpha t}}$$

$$\Rightarrow \boxed{\underbrace{\begin{bmatrix} -\alpha^2+\omega_0^2+\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2+\omega_0^2+\omega^2 \end{bmatrix}}_{\equiv \underline{P}} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

1(d) Det $(\underline{P}) = 0$.

$$\Rightarrow (-\alpha^2+\omega_0^2+\omega^2)^2 - \omega^4 = 0$$

$$\Rightarrow -\alpha^2+\omega_0^2+\omega^2 = \pm\omega^2$$

Case ①: If ⊕ : $-\alpha_1^2 = -\omega_0^2 \Rightarrow \alpha_{1,\pm} = \pm\omega_0$

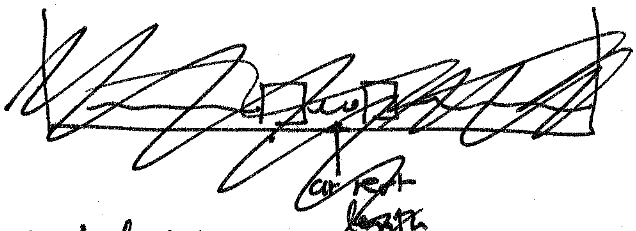
Case ②: If ⊖ : $-\alpha_2^2 = -2\omega^2-\omega_0^2$

$$\rightarrow \boxed{\alpha_{2,\pm} = \sqrt{\omega_0^2+2\omega^2}}$$

$|\alpha_{1,\pm}|$ and $|\alpha_{2,\pm}|$ are the 2 normal angular frequencies.

(e1) · Normal mode w/ $\alpha_1 = \omega_0$:

describes symmetric $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ motion of 2 blocks ;



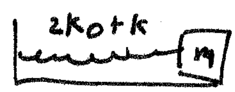
$\Rightarrow$ $x_1(t) = x_2(t)$ at all times

$\Rightarrow$ middle spring always remains at its rest length

This is equivalent to one block of mass m, attached to spring k. 

· Normal mode w/ $\alpha_2 = \omega_1$: describes antisymmetric $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

motion of 2 blocks : $\Rightarrow$ $x_1(t) = -x_2(t)$ at all times.

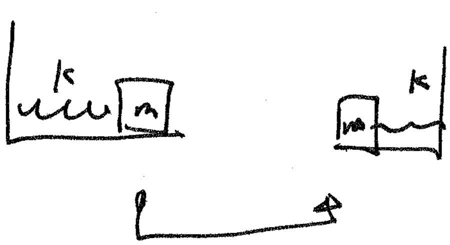
This is equivalent to one block of mass m, attached to spring $2k_0 + k$. : 

(f) $A_2 = 0$. $\Rightarrow$ $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_1)$.

Since $x_1(t) = x_2(t)$ at all times, middle spring is never compressed or stretched. $\Rightarrow$ $\boxed{\text{No energy stored in middle spring.} \Rightarrow \text{No energy is lost.}}$

when middle spring is cut.

After middle spring is cut :



both oscillate still with frequency $\omega_0 = \sqrt{k/m}$.

as if nothing happened. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_1)$ afterwards

[g] $\underline{A_1 = 0}$ $A_2 > 0$.

At the instant that $X_1(t) = -A_2$

$\Rightarrow X_2(t) = A_2$.

$\Rightarrow$ Middle spring is stretched by $\underline{2A_2}$.

$$\Rightarrow \quad PE_{\substack{\text{middle}\\ \text{spring}}} = \frac{1}{2}(2A_2)^2 K_0$$

$$= \boxed{2A_2^2 K_0}$$

$\uparrow$ This energy is lost.

(8)