MITES 2008 : Physics III - Oscillations and Waves :: Problem Set 2.

Massachusetts Institute of Technology
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(Due on Wednesday, July 9, 2008 at 11:59 PM: Slip under Louis' door in Simmons.)

What this problem set is about:
This problem set is designed to help you shape your understanding of forced oscillations of a single particle, and coupled oscillations of more than one particle. Starting from coupled oscillations of two or three particles, then extending to many more particles, we'll be able to *derive* the *wave equation*. That is what we will do next week, after our midterm examination.

Problem 1. Ring oscillator.

Two beads, each of mass m, are threaded through a rigid ring backbone. The backbone is well-oiled so there is no friction as the beads slide around this circular ring of radius R. They can only move around in the circle. The beads are attached to each other via two identical Hookian springs, each with spring constant k. The entire system is resting on a table top, so we can ignore gravity in this problem. Let's find out what sorts of motion is possible for these two beads by deriving their equations of motion, then solving them. Fig. 1 is a depiction of our system.

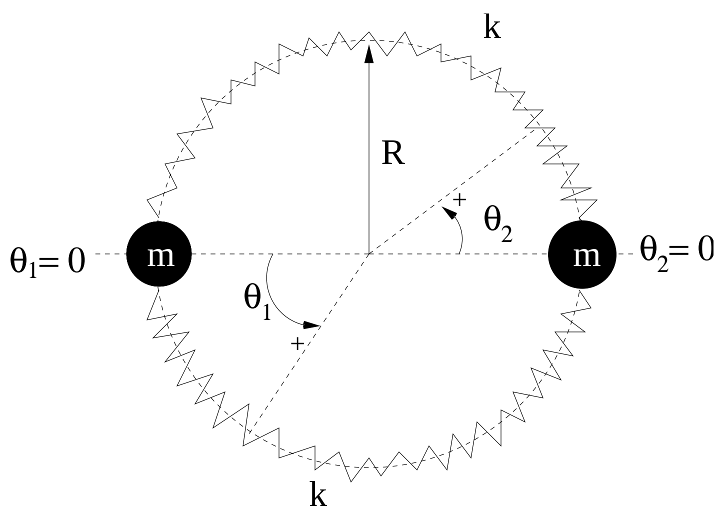

FIG. 1: The "Ring oscillator".

(a.) Draw what the system would look like in equilibrium.

(b.) Set up the coordinate system so that θ_1 and θ_2 are the angular positions of the two particles respectively, relative to their equilibrium positions. θ_1 is set to zero where particle 1 is in equilibrium, and θ_2 is set to zero where particle 2 is in equilibrium. For this problem, let's set up the sign convention so that θ_1 and θ_2 are both positive when the two beads are moving around counter-clockwise, negative if the two beads are moving clockwise around the ring.
Using this coordinate system, derive the equations of motion describing the two particles by using Newton's second law. Notice that the springs will always exert a *tangential* force on the beads. That is, the force vector is tangent to the circle right at the point where the bead of interest is located. Notice that since there are two particles, you should get two separate equations of motion.

(c.) Next, solve the equations of motion you derived in (b.) by adding up the two equations of motion, and also subtracting the two equations of motion. You'll obtain an "effective" equation of motion when you add up the two equations (describing $\theta_1 + \theta_2$), and another equation of motion when you subtract the two equations (describing θ_1-θ_2). What sort of motion does each of these two **normal coordinates** describe? Answer this question by solving the two "effective" equations of motion for each of these two normal coordinates. Then explain what motion your solutions are depicting by using pictures inspired from these solutions.

(d.) Write down the general solution for $\theta_1(t)$ and $\theta_2(t)$ in terms of these two normal coordinates. Be sure to check that your expressions for θ_1 and θ_2 each have suitable number of *free parameters.*

Problem 2. Toy model of a triatomic molecule: CO_2

In this problem, we model the longitudinal vibrational modes of the triatomic molecule, CO_2, using springs connecting three masses as shown in Fig. 2.

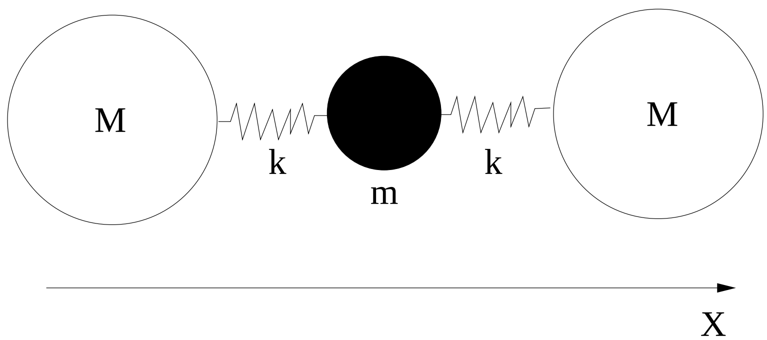

FIG. 2: Classical model of the longitudinal vibrational modes of CO_2. M is the mass of oxygen atom, and m is the mass of carbon atom. The springs represent a toy model of the valence bond between the C and O, with k being the characteristic "strength" of the bond.

(a.) Set up a coordinate system describing the displacement of each of the three atoms about their equilibrium positions along the x-axis shown in Fig. 2. You may use x_1, x_2, and x_3 to denote these displacements. In this problem, we will only consider the motion of the three atoms along the x-axis : longitudinal vibrations and translations.

(b.) Derive the equations of motion for each of the three atoms. Notice that you'll get three equations of motion, one for each atom.

(c.) Put the equation of motion in a matrix form: $P \cdot x$. The final form of the equations should be in the following compact form:

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \\ \frac{d^2x_3}{dt^2} \end{pmatrix} \tag{1}$$

where the entries of the matrix p_{ij} are for you to determine.

(d.) Let's solve the complex-equivalent EOM in the matrix form. That is, we want to solve:

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{d^2z_1}{dt^2} \\ \frac{d^2z_2}{dt^2} \\ \frac{d^2z_3}{dt^2} \end{pmatrix} \tag{2}$$

where $z_j(t)$ is now the complex-equivalent of $x_j(t)$. To solve this, we try our beloved "guess and check" method. In this case, our guess is

$$\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} exp(i\alpha t) \tag{3}$$

where A, B, and C are arbitrary constants and α is as of yet undetermined parameter. By plugging this guess into Eq'n (2), show that our guess (Eq'n (3)) is indeed a solution to the complex-equivalent EOM (2) and as long as α satisfies certain relationship that you derive in this process. It may help you to use the following formula for the determinant of a 3 x 3 matrix:

$$Det(P) = p_{11} Det \begin{pmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{pmatrix} - p_{12} Det \begin{pmatrix} p_{21} & p_{23} \\ p_{31} & p_{33} \end{pmatrix} + p_{13} Det \begin{pmatrix} p_{21} & p_{22} \\ p_{31} & p_{32} \end{pmatrix} \tag{4}$$

(e.) The α's you've found in (d) are called the **normal angular frequencies** of the CO_2's longitudinal vibrations. But one of them really isn't a vibrational mode. Which one is it? What motion does that normal angular frequency describe?

(f.) For *each* of the normal angular frequencies you found in (e), find the relationships between A, B, and C (the constants in your guess in Eq'n (2)). From this, describe using pictures what kind of vibration each of the normal angular frequencies describe.

(g.) Finally, write down the *general* solution $z(t)$ (in complex number form) in terms of all the normal modes you found, using the relationships between various constants you found in (f.).

Problem 3. Simple harmonic motion with sinusoidal external force.

In class, we looked at an equation of motion describing a system with a damping force, restoring force, and an external sinusoidal driving force $F(t) = F_0 cos(\omega t)$, where ω is the angular frequency of the driving force $F(t)$. From class (and in your lecture notes), we found that the equation of motion describing the motion of such a particle is (you should know how to derive this):

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} cos(\omega t), \tag{5}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency associated with the spring (spring constant k) and the particle of mass m, and $\gamma = \frac{b}{2m}$ represents the strength of damping (b is the damping constant). These terms were all defined in our class notes. In this problem, you're asked to reproduce the calculations in the notes that we rushed through in class due to lack of time.

(a,) Write down the complex-equivalent equation of motion corresponding to equation (5).

(b.) Let's guess a solution to the complex-equivalent equation of motion (EOM) to be $z_p(t) = Aexp(i\omega t)$. Show that this is indeed a solution of the EOM provided that A obeys a relationship you derive in the process.

(c.) $z_p(t)$ is only a *particular* solution to the equation of motion. State what we mean by the *linearity* (aka. *superposition principle*) property of the EOM (*Hint*: What does it mean for a differential equation to be *linear*)? Using this linearity principle, write down the *general solution* to the complex-equivalent equation of motion.

(d.) Now, step-by-step, when the system is *underdamped*, show that the *real* part of the general solution you found in (c.) becomes:

$$x(t) = Re(z(t)) = Ce^{-\gamma t} cos(\tilde{\omega} t - \phi) + \frac{F_0}{m} \frac{(\Omega^2 cos(\omega t) + 2\gamma\omega sin(\omega t))}{(\Omega^4 + 4\gamma^2\omega^2)}, \tag{6}$$

where $\Omega \equiv \sqrt{\omega_0^2 - \omega^2}$. Notice that $x(t)$ is the *actual* physically meaningful (because it's a real numbered function) position of the particle. What do the terms C, ϕ, Ω, and $\tilde{\omega}$ mean?

(e.) The first term of $x(t)$ you found in (d) is made up of two parts: the damped part and the part dealing with the driving force. What happens to the damped part of $x(t)$ in the limit of long time? That is, what happens when $t \gg \frac{1}{\gamma}$? Thus, after a long enough time (which isn't really all that long given that γ is typically large enough for realistic damped systems) has passed, how does the position of the block $x(t)$ behave?

(f.) The damped part of $x(t)$ is called the **transient part** of the particle's motion precisely due to the behavior you found in (e). It becomes irrelevant after a while, and the particle thus "loses" any memory of its history. That is, two particles, each starting with a different value of C and ϕ, end up behaving the same way since the only place in $x(t)$ that the free parameters C and ϕ come in is in the transient part. Say we define:

$$x_p(t) = \frac{F_0}{m} \frac{(\Omega^2 cos(\omega t) + 2\gamma\omega sin(\omega t))}{(\Omega^4 + 4\gamma^2\omega^2)} \tag{7}$$

which is the remaining part of $x(t)$. For what value(s) of the driving angular frequency ω is the amplitude of $x_p(t)$ maximum? This frequency is called the *resonant frequency.*

Problem 4. Quenching unwanted vibrations.

A simple engineering principle for quenching unwanted vibrations is by attaching a small mass m to a larger mass M through a spring, whose Hookian spring constant is k. Suppose that a sinusoidal force $F(t) = F_0 cos(\omega t)$ is applied to the larger object. Show that if the driving angular frequency ω matches the natural angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$, the larger object (of mass M) does not move at all. Hence, if you want to stop a particle from having oscillations when driven by a sinusoidal force of angular frequency ω, then you just have to "load" the object with an additional mass m and spring of stiffness k. (Note: Assume that all motion is confined along the x-axis in this problem.)

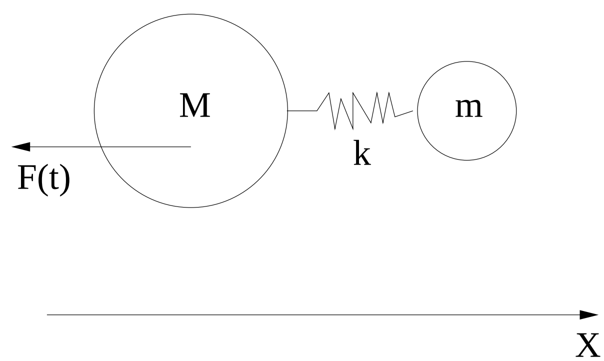

FIG. 3: "Silencing" unwanted vibrational modes.