MITES 2008 : Physics III - Oscillations and Waves :: Problem Set 3.

Massachusetts Institute of Technology
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Recitation Instructor: Louis Fouche
(Due on Wednesday, July 16, 2008 at 11:59 PM , slip under Louis' door in Simmons.)

What this problem set is about:
Deriving the wave equation by taking the *continuum limit.* Start from Newton's second law describing a single particle, then combining the equation of motion for every single particle, we obtain a description of the **collective beahvior** of a large number of coupled oscillators. Wave is simply an excitation of these collective modes.

Problem 1. Sound waves (Phonons) in a one-dimensional crystal.

In class we looked at a one dimensional chain of N identical atoms, each with mass m. In this system, each atom was bound to its two nearest neighbors (left and right atoms) by identical Hookian springs with spring constant k. In equilibrium, the distance between one atom and its neighboring atom is a (called *lattice constant*). The entire chain of atoms has a fixed length L. Let y_j denote the displacement of the j-th atom in the chain from its equilibrium position. Vibrational modes supported by this media are good examples of **collective excitation modes** that arise from interaction of many particles. The train of compressions and rarefaction traveling down the "slinky" represented by our chain of atoms, is called a **phonon**. It is often called "sound wave".

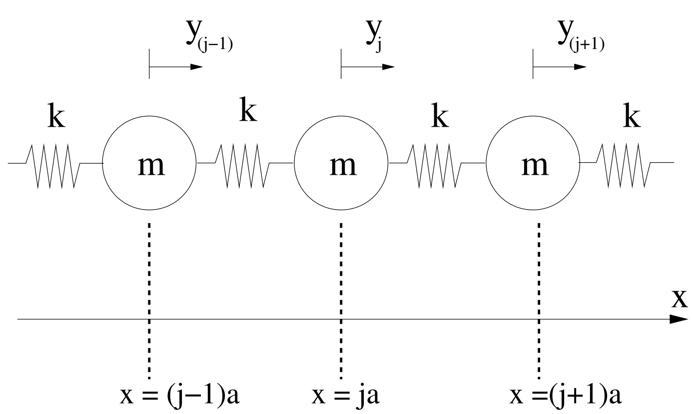

FIG. 1: One dimensional crystal.

(a.) Assume that N is very large and that a is very small. Show that in this limit, the total energy of this one-dimensional crystal is

$$E_{tot} = \int_0^L \frac{\rho}{2}(\frac{\partial y}{\partial t})^2 dx + \int_0^L \frac{D}{2}(\frac{\partial y}{\partial x})^2 dx \quad (1)$$

where D and ρ are constants you need to determine in terms of the parameters given in this problem. What is the physical meaning of both of these constants? To do this, start by writing a *discrete* sum of individual spring and particle energies, then you need to turn the discrete sum into integrals by taking appropriate limits. Be sure to justify what it means for N to be very large, and a to be very small (i.e. "large" compared to what? "small" compared to what?).

(b.) As was done in class, derive the wave equation ("Equation of motion for a wave supported by this crystal") by starting from Newton's second law applied to a particular atom (j-th atom in the lattice). Be sure to state what *continuum limit* you're taking to turn the finite difference equations of motion into differential equations in your derivation, and justify each step, making sure to specify what approximations you're using.

(c.) Solve the wave equation and thus state the general solution.

Problem 2. Loaded string.

Consider the loaded string example in class. Since this is an important example of how one can go from a "discrete" picture (rigid (massless) rods connecting N atoms) to a "continuous" picture (smooth (uniform mass density) string supporting transverse wave), you're being asked to rederive the wave equation and the associated normal modes in this problem. This is merely a reproduction of calculations we did in class, but an important exercise for you to reinforce the techniques used in this procedure.

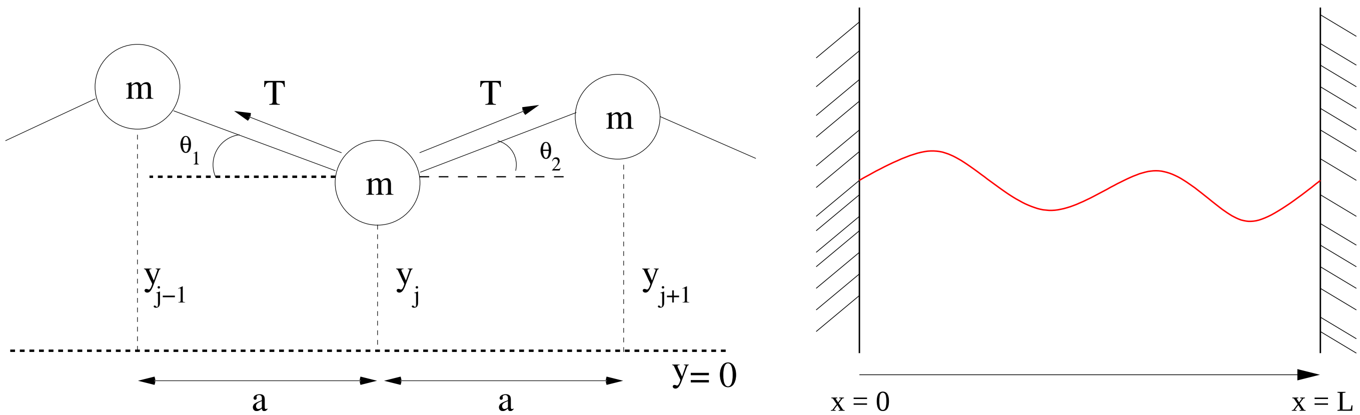

FIG. 2: Loaded string: Rigid rods, each under constant tension T, connects one atom to another in this "necklace". Both ends of the necklace are fixed to a wall. Zooming out (i.e. in the *continuum limit*, we would see the figure on the right.

(a.) By starting from Newton's second law applied to the j-th particle in this "necklace" of atoms, show that its equation of motion is

$$\frac{d^2 y_j}{dt^2} = \frac{T}{ma}(y_{j-1} - 2y_j + y_{j+1}) \tag{2}$$

where $y_j(t)$ is the vertical displacement of the j-th particle from its equilibrium y_j=0 position. Also show that the net horizontal component force on each atom is zero. Since we're considering the ends of this "necklace" to be fixed (glued to a wall), what is the equation of motion for j=1 and j=N? Apply an appropriate *boundary condition* to get these two particular equations of motion.

(b.) Show that in the *continuum limit* (need to rigorously specify what this means, as you did in problem 1), above equation becomes the wave equation:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}, \tag{3}$$

where v is the "wave speed" you need to figure out in terms of the parameters given in this problem.

(c.) Write down the *complex-equivalent* equation of motion for the j-th particle (i.e. complex-equivalent of Equation in (a)). Then, guess a solution of the form: $y_j(t) = A_j exp(i\omega t)$, where ω is a value yet to be specified and A_j is a constant (but can be different for different values of j, thus the subscript). Check that this is a solution describing the position of the j-th particle provided that the following condition is satisfied:

$$-A_{j-1} + (2 - \frac{ma\omega^2}{T})A_j - A_{j+1} = 0 \tag{4}$$

Since both ends of the system are fixed, what should the values of A_0 and A_{N+1} be?

(d.) By rearranging above equation, we can rewrite it as

$$\frac{A_{j-1} + A_{j+1}}{A_j} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}, \tag{5}$$

where $\omega_0^2 = \frac{T}{ma}$. Now, let's solve the equation. This is again by a guess. Let's assume that we can express the amplitude of jth particle as $A_j = Cexp(ij\theta)$, where C and θ are constants. By plugging this into above equation and invoking two *boundary conditions* (state what these are), show that $A_j = Cexp(ij\theta)$ is a solution as long as the following conditions hold:

$$Csin(j\theta) = 0 \qquad \text{(when j = 0)} \tag{6}$$

$$Csin((N+1)\theta) = 0 \tag{7}$$

(e.) Hence, show that $A_j = Csin(\frac{jn\pi}{N+1})$ where $n = 1, 2, ..., N$. What are the normal mode angular frequencies ω_n? How does this compare to standing waves you learned about in recitation?