MITES 2008 : Physics III - Oscillations and Waves :: Problem Set 4.

Massachusetts Institute of Technology
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(Due on Wednesday, July 23, 2008 at 11:59 PM , slip under Louis' door in Simmons.)

What this problem set is about:
Fourier series as a tool for expressing a general wave in terms of normal modes. Solving wave equation subject to various boundary conditions. Working with "complex number form" of waves. Dispersion relation describing energy of collective vibrations in a crystal.

Problem 1. Expanding wall.

(a.) In class, we learned that the normal modes for a vibrating string with its ends fixed (at $x = 0$ and $x = L$) at two walls that are separated by distance L is

$$y_n(x,t) = \{A_n sin(\omega_n t) + B_n cos(\omega_n t)\} sin\Big(\frac{n\pi x}{L}\Big), \tag{1}$$

where $n = 1, 2, 3....$ Since this is an important example of solving the wave equation using a boundary condition, derive this solution again by starting with the *complex-valued* wave equation

$$\frac{\partial^2 \tilde{y}}{\partial t^2} = v^2 \frac{\partial^2 \tilde{y}}{\partial x^2}, \tag{2}$$

where $\tilde{y}(x,t)$ is a complex-valued solution to the wave equation. To solve this, guess the solution to be $\tilde{y}(x,t) = \{Aexp(ikx) + Bexp(-ikx)\}exp(-i\omega t)$, then impose the two boundary conditions on it (one for each wall). What do $exp(ikx)$ and $exp(-ikx)$ represent physically?

(b.) Suppose that at $t = 0$, the string has the following shape, described by $\phi(x)$ (for $0 < x < L$):

$$\phi(x) = -\alpha\{\frac{L^2}{4} + (x - \frac{L}{2})^2\} \tag{3}$$

and is initially at rest (i.e. you're holding the string in the shape described by $\phi(x)$ at $t = 0$. Express $y(x, t = 0)$ as a linear combination of the normal modes:

$$y(x, t = 0) = \sum_{n=1}^{\infty} c_n sin(\frac{n\pi x}{L}), \tag{4}$$

where you need to determine the **Fourier coefficients** c_n.

(c.) Immediately after $t = 0$, you release the string so that it now changes its shape from $y(x, 0) = \phi(x)$. Write down the expression describing the subsequent shape of string, $y(x, t)$.

(d.) Suppose that we have the same initial condition (string initially held at rest by hand, with initial shape described by $\phi(x)$.) but now the distance between the wall has been extended to $2L$. What are the new normal modes and what is $y(x, t)$ for $t > 0$ when you let go of the string? Notice that now

$$\begin{aligned}\phi(x) &= -\alpha\{\frac{L^2}{4} + (x - \frac{L}{2})^2\} && \text{(for } 0 < x < L) \\ &= 0 && \text{(for } L < x < 2L)\end{aligned} \tag{5}$$

Problem 2. Characteristic Impedance of a String (String viewed as a collection of forced SHOs.)

Impedance "Z" is a measure of resistance experience by a forced simple harmonic oscillator (SHO), whether it is a mechanical forced SHO (e.g. block-spring driven by a sinusoidal force), or an electrical forced SHO (e.g. RLC-circuit). In this problem, we first figure out how we measure the impedance of a transverse wave. Then we will use this information in problem 3 to figure out how transverse waves penetrate (transmitted) and bounce back (reflected) at a boundary between two different media.

First, impedance Z is defined as

$$Z = \frac{F_0}{v_0} \qquad \text{(Impedance)} \tag{6}$$

where F_0 is the maximum force that the forced SHO is subject to, and v_0 is the maximum speed that the forced SHO reaches in its oscillations. You can think of why this is a measure of *resistance* experienced by the oscillator if you think back to *Ohm's law* in electrical circuits, which says that $R = \frac{V}{I}$, where R (like our Z) is the electrical resistance, V (like our F_0, some measure of "force") is the voltage across a circuit element, and I (like our v_0, some measure of "speed") is the electrical current.

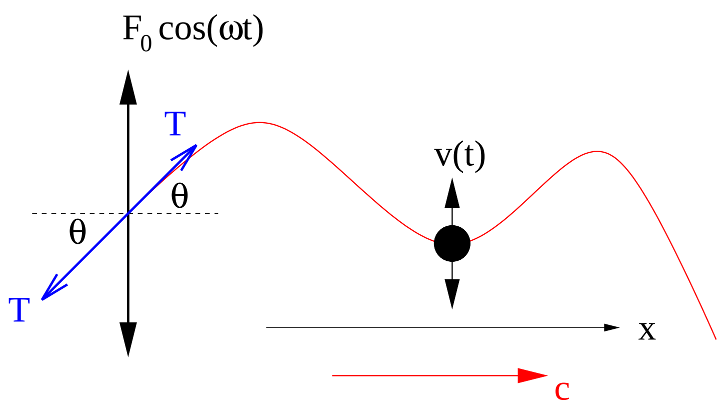

FIG. 1: Transverse wave with wave speed c. Each atom making up the string simple harmonically oscillates vertically with velocity $v(t)$. The string is under constant tension T throughout its length. Each particle at position x can be considered to be a forced oscillator subject to sinusoidal force of $F(t) = F_0 cos(kx - \omega t)$

Imagine a transverse wave, traveling along the x-axis with wave speed c. Recall that for a transverse wave, the wave travels horizontally down the string, while the individual atoms making up the string simple harmonically oscillates vertically up and down. Suppose the wave is described by $y(x,t) = Asin(kx - \omega t)$. Which direction is this wave moving (right or left)? By looking at above figure and applying Newton's second law to the particle at position x, show that the impedance Z is

$$Z \equiv \frac{F_0}{v_0} = \frac{T}{c} = \rho c \tag{7}$$

where T and ρ are the constant tension in the string and the uniform density of string respectively. (*Hint*: Assume that vibrations $y(x,t)$ are small so that θ is a small angle ($|\theta| \ll 1$. In this case, $cos(\theta) \approx 1$.)

Problem 3. Transmission and Reflection of a Transverse Wave at a Boundary Between Two Media.

In problem 2, we learned that impedance is a material property of the string (or any material that the wave is traveling in, for that matter). Then what would happen if one wave has to go from one type of media to another? For example, if we join two strings made of two different materials, what would happen to a transverse wave as it passes from one string to another?

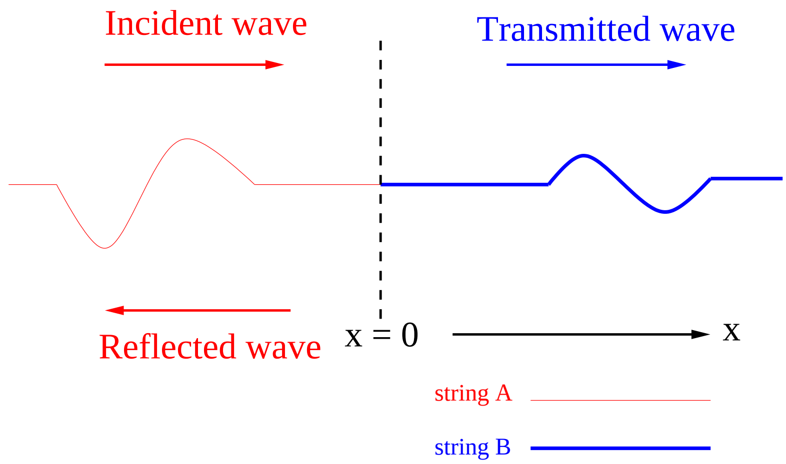

FIG. 2: Incident and reflected waves travel in string A, while transmitted wave travels in string B. String A and B have mass density ρ_1 and ρ_2 respectively. Both are under constant tension T. In string A, wave travels at speed c_1 while in string B, wave travels at speed c_2.

Consider the set up shown in above figure. An incident wave, traveling to the right is moving in string A. Some of this wave is reflected at the boundary (x=0) between the two strings and travels back to the left in string A, while some fraction of the incident wave is transmitted and thus moves to the right in string B. We wish to find out what fraction of the incident wave gets transmitted to string B. String A and B have mass density ρ_1 and ρ_2 respectively. Both are under constant tension T. In string A, wave travels at speed c_1 while in string B, wave travels at speed c_2.

(a.) We carry out our analysis using complex numbers. Recall that linearity of wave equation affords us this luxury (remember that complex numbers simplify our calculations greatly).
(i.) Write down an expression for a **plane wave** in complex number form, that represents the incident wave.
(ii.) Write down an expression for a **plane wave** in complex number form, that represents the reflected wave.
(iii.) Write down an expression for a **plane wave** in complex number form, that represents the transmitted wave.

(*Hint:* A *plane wave*, in real number form, refers to just $Acos(\pm kx - \omega t) + Bsin(\pm kx - \omega t)$. Thus, the complex version of a plane wave is just $Cexp(i(\pm kx - \omega t))$. In this problem, you need to pick the right sign in front of the wave number to indicate which direction each plane wave is traveling.)

(b.) Using (a.), write down an expression (still in complex number form) describing the shape of the **resultant wave** in string A, and also an expression describing the shape of resultant wave in string B. Call y_A and y_B the shape of resultant wave in A and B respectively. (*Hint*: Use superposition principle).

(c.) If we had joined the two strings at some arbitrary x (instead of $x = 0$), what would the *boundary conditions* at that x be? You should have two boundary conditions (so two equations, one for each boundary condition). By thinking about what sort of restrictions are placed the shape of wave at the junction of the two strings, you can come up with these boundary conditions. (Think of smoothness (no kinks), and the fact that the two strings are always joined to each other).

(d.) Now, let's say that the two strings are jointed at a particular position $x = 0$. Apply the two boundary conditions you found in (c.) to $x = 0$. From these two equations, show that the **reflection coefficient R**, defined as $R = \dfrac{|A_{reflect}|^2}{|A_{incident}|^2}$, is

$$R = \frac{|A_{reflect}|^2}{|A_{incident}|^2} = \Big|\frac{Z_A - Z_B}{Z_A + Z_B}\Big|^2, \tag{8}$$

where $A_{reflect}$ and $A_{incident}$ are the (complex) amplitudes of reflected and incident waves respectively. Z_A and Z_B are the impedances in string A and B respectively. Also show that the **transmission coefficient T**, defined as $T = \Big|\frac{A_{transmitted}}{A_{incident}}\Big|^2$, is

$$T = \Big|\frac{A_{transmitted}}{A_{incident}}\Big|^2 = \Big|\frac{2Z_A}{Z_A + Z_B}\Big|^2, \tag{9}$$

where $A_{transmitted}$ is the (complex) amplitude of transmitted wave. Notice that $\Big|A\Big|$ is the *modulus* of the complex number A.

Notice that the reflection and transmission coefficients are just material properties, independent of ω. In the case of transverse waves generated on a string, this means that no matter with what frequency your hand holding the left end of string A oscillates up and down (thus generating an incident transverse wave with angular frequency equal to that of your simple harmonically oscillating hand), the fraction of transmitted and reflected waves will remain unchanged.

(e.) Now, let's suppose that the impedance of string B is infinity ($Z_B = \infty$). What are now the reflection and transmission coefficients? Also, write down the expression for incident, reflected, and transmitted waves in their complex form for this special case. What does this situation describe physically? What is the **phase shift** upon reflection at the boundary in this case (i.e. what is the relative phase difference between incident and reflected waves)?

Problem 4. Transmission and Reflection of Energy.

(a.) Consider a transverse wave moving down a string of uniform mass density ρ (mass / length). Thinking of each atom making up the string as a simple harmonic oscillator (as we did in problem 2), show that the energy per unit length ρ_E of a transverse wave traveling at speed c with angular frequency ω and amplitude A is

$$\rho_E = \frac{1}{2}\rho\omega^2 A^2 \tag{10}$$

(b.) Now, show that the *power* P transmitted through the wave is

$$P = \frac{1}{2}\rho\omega^2 A^2 c \tag{11}$$

(*Hint*: It will help you to recall what the definition of *power* is in mechanics.)

(c.) Let's consider the two strings joined at $x = 0$ in problem 3. What can you say about energy that is transported to the boundary by the incident wave, compared to the energy carried away by both the reflected and transmitted waves? Using parameters given in problem 3 and in your answers to problem 3, show that energy is conserved. To start this off, show that the rate at which energy arrives at the boundary with the incident wave is

$$\frac{1}{2}Z_A\omega^2(A_{incident})^2 \tag{12}$$

(d.) Show that the fraction of reflected energy compared to incident energy is

$$\Big(\frac{Z_A - Z_B}{Z_A + Z_B}\Big)^2 \tag{13}$$

Also, show that the fraction of transmitted energy compared to incident energy is

$$\frac{4Z_A Z_B}{(Z_A + Z_B)^2} \tag{14}$$

Problem 5. Anti-glare coating on lenses (Eg. of Impedance matching).

Eye glasses and other lenses where some portion of incident wave (light) has to be transmitted through the lens (so that you can see through the lens) while eliminating reflection from the lens (so that your friends can see your eyes, through your eye glasses), use a technique called **impedance matching**. To do this, an "anti-glare coating" of thickness $\frac{\lambda}{4}$ is placed on top of the lens, where λ is the wavelength of electromagnetic radiation that you *do not* want to be reflected off. In this problem, we use an example of transverse waves going through three different types of strings and analysis not much more complicated than what we used in problem 3 to investigate why a thickness of $\frac{\lambda}{4}$ can do the job.

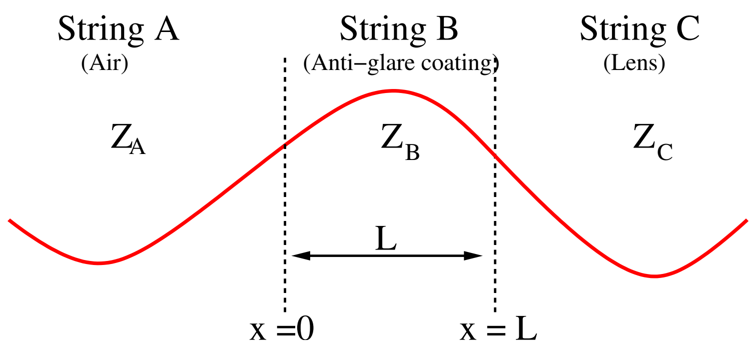

FIG. 3: Three different media. We use three strings, each made from different material, to model electromagnetic wave going through the lens from air. String A ($x < 0$) has impedance Z_A, wave speed c_1 and uniform mass density ρ_1. String B ($0 < x < L$) has impedance Z_B, wave speed c_2 and uniform mass density ρ_2. String C ($L < x$) has impedance Z_C, wave speed c_3 and uniform mass density ρ_3. All three strings are under the same constant tension T.

The fact that electromagnetic wave (as we will learn in lectures soon) is a transverse wave obeying the same wave equation (but with different physical meaning) allows us to model the anti-glare coating on a lens using transverse waves on strings. To do this, we use three strings, each made from different material. String A ($x < 0$) has impedance Z_A, wave speed c_1 and uniform mass density ρ_1. String B ($0 < x < L$) has impedance Z_B, wave speed c_2 and uniform mass density ρ_2. String C ($L < x$) has impedance Z_C, wave speed c_3 and uniform mass density ρ_3. All three strings are under the same constant tension T.

(a.) Consider your hand holding the left end of string A simple harmonically oscillating up and down with angular frequency ω. This generates an incident wave of angular frequency ω traveling along string A to the right. As we did in part (a) of problem 3, write down the expressions for the incident, reflected, and transmitted waves. Note that in string A, there is an incident and a reflected wave. In string B, there is a reflected and transmitted wave. In string C, there is only a transmitted wave.

(b.) As you did in problem 3 (c), write down the boundary conditions at each of the two boundaries ($x = 0$ and $x = L$).

(c.) By using the boundary conditions above, derive the following relation:

$$\frac{\text{Transmitted energy in string C}}{\text{Incident Energy}} = \frac{4r_{AC}}{(r_{AC}+1)^2 cos^2(k_B L) + (r_{AB} + r_{BC})^2 sin^2(k_B L)}, \tag{15}$$

where $r_{AB} \equiv \frac{Z_A}{Z_B}$, $r_{BC} \equiv \frac{Z_B}{Z_C}$, and $r_{AC} \equiv \frac{Z_A}{Z_C}$. k_B is the wave number of the transverse wave in string B.

(d.) By picking the thickness of the middle medium (anti-glare coating) to be $L = \frac{\lambda_B}{4}$, and by making $Z_B = \sqrt{Z_A Z_C}$, show that all energy is transmitted through the middle, into string C. Here, λ_B is the wavelength in string B.

This shows that indeed, when we *match* the impedance of the material in the middle (string B) to be the *geometric mean* of the two media that it abuts, then no reflection occurs. Notice that the wave in string B is a standing wave. This is how anti-glare coating works. SImilarly, electrical transmission lines are impedance matched by putting in the right type of wires with length $\frac{\lambda}{4}$ (λ here being the wavelength of the AC current going through) so that energy loss is minimized (It would be wasteful to sent all that current, only to have some fraction of it come back to the generating station).

Problem 6. One dimensional crystal made up of two kinds of ions.

NaCl is a three dimensional crystal, with a cubic structure (See http://en.wikipedia.org/wiki/Nacl). In this problem, we consider the simpler, one-dimensional version of this. It turns out that many of the essential physics of the 3D crystal is captured in our 1D toy model.

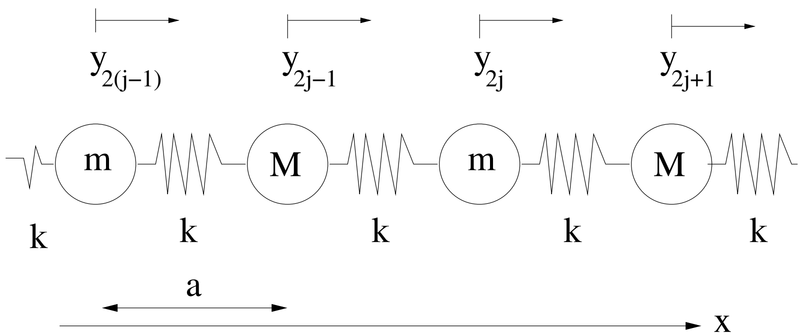

FIG. 4: One dimensional NaCl: ionic crystal. Na^+ ions and Cl^- ions alternate.

Imagine a one dimensional "slinky", with the smaller Na^+ ions occupying positions $...x_{2j-1}, x_{2j+1}, x_{2j+3}$ and so on, while the larger Cl^- ions occupying positions $...x_{2j-2}, x_{2j}, x_{2j+4}$ and so on. M is mass of Na^+ while m is the mass of Cl^-. Therefore, we have a periodic chain of alternating Na^+ ions and Cl^- ions. The equilibrium distance between two adjacent ions is a. In this problem, we study what happens when these ions vibrate about their equilibrium positions. Let y_j denote the small displacement of the jth ion from its equilibrium position.

(a.) By writing down the Newton's second law, show that the equations of motion describing Na^+ and Cl^- ions are

$$m\frac{d^2 y_{2j}}{dt^2} = \frac{T}{a}(y_{2j+1} + y_{2j-1} - 2y_{2j})$$
$$M\frac{d^2 y_{2j+1}}{dt^2} = \frac{T}{a}(y_{2j+2} + y_{2j} - 2y_{2j+1})$$

where T is a parameter you need to determine in terms of given parameters.

(b.) Let's solve the complex-number equivalent equations of motion. By writing down the complex-number equivalent EOMs, and then guessing our normal modes to be

$$\begin{aligned} \tilde{y}_{2j} &= A_m exp(i(\omega t - 2jka)) \\ \tilde{y}_{2j+1} &= A_M exp(i(\omega t - (2j+1)ka)) \end{aligned} \tag{16}$$

show that our guess is indeed correct provided that the normal mode angular frequency ω obeys

$$\omega^2 = \frac{T}{a}\Big(\frac{1}{m} + \frac{1}{M}\Big) \pm \frac{T}{a}\Big\{(\frac{1}{m} + \frac{1}{M})^2 - \frac{4sin^2(ka)}{mM}\Big\}^{1/2} \tag{17}$$

These are the normal mode angular frequencies. $\omega(k)$ is also called **dispersion relationship**.

(c.) Notice that ω is a function of the wave number k. Keeping in mind that $m > M$, plot $\omega(k)$ as a function of k when you take the positive root in above equation. Also, on the same graph, plot $\omega(k)$ as a function of k when you take the negative root in the above equation. Plot both for the k-range of $[0, \frac{\pi}{2a}]$. What expression does the disperson relation you found in (b) simplify to in each of the two cases?

(d.) You should have found two "branches" in your plot in (c.). The upper branch (curve) is called the **optical branch** while the lower one is called **acoustical branch**. Provide a sketch and comment what type of wave is formed in the optical mode, and also the type of wave formed in the acoustical mode. It may help you to consider what happens in the optical mode and in the acoustical mode when the wavelength is long (thus small k).

The optical mode is the mode of vibration generated when an electromagnetic wave (transverse) hits a crystal formed by ions of opposite charges. This lends the name "optical mode". Acoustical mode is similar to the sound waves (longitudinal). It turns out that it's only a short step from here to calculation of specific heat of the crystal.