MITES 2008 : Physics III - Oscillations and Waves :: Problem Set 5
(Final Exam Review Problem Set)

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(Due on Monday, July 28, 2008 at 1:20 PM in class.)

What this problem set is about:
This problem set is designed to test your understanding of the key concepts covered in our course. It is a **review problem set**, intended to help you review important concepts we covered in the course in preparation for your final examination on Wednesday, July 30, 2008.

Note about Grading: Only **Problem 6: EM waves** will be graded. But you're **strongly** recommended to solve the other problems, and understand the **procedure** for solving them fully. The solution set for these problems will be handed out to you on the due date. You'll do very poorly on the final exam if you just "memorize" the solutions to these problems by reading the solution set without even attempting to solve them yourself.

Problem 1. Simple Harmonic Oscillator.

Consider a block of mass m attached to a Hookian spring with stiffness k on an inclined plane as shown in Figure 1. One end of the spring is fastened to a wall at the top of the ramp while the other end is attached to the block. The angle of inclination is θ. The *position* of the block (not its displacement) measured relative to the wall, and parallel to the incline, is s. $s = 0$ at the wall. Let $s = s_0$ be the equilibrium position of the block. Acceleration due to gravity is a constant g. Assume that the incline is *frictionless*.

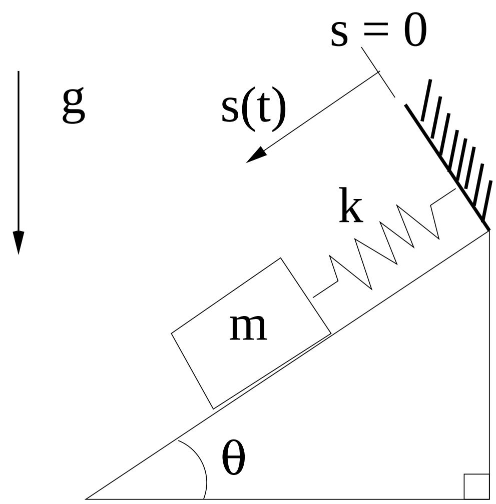

FIG. 1: A block attached to a spring on a *frictionless* inclined plane. Spring has stiffness k and one of its ends is fastened to a wall at the top of the inclined ramp. The ramp is inclined with an angle θ. The block has mass m and the acceleration due to gravity is a constant g. The position of the block, measured relative to the wall and down along the ramp is s. $s = 0$ at the wall. Let $s = s_0$ be the equilibrium position of the block.

(a.) Assuming that the rest length (aka. natural length) of the spring is zero, what is the equilibrium position s_0 of the block? Answer in terms of m, g, k, and θ.

(b.) Write down the equation of motion describing the block, thus showing that the block acts as a simple harmonic oscillator. What is the natural angular frequency ω_0 of the block?

(c.) Write down the general solution to the equation of motion you derived in (b.). Define what the physical meaning of all your free parameters in your solution are. (When you're told to "write down", it means state, not derive. Of course, you can derive the general solution by the "guess and check" method in this case.)

(d.) Write down the total energy of the system above. The system consists of the block and the spring so the total energy should be coming from both of these objects.

(e.) From the total energy, derive the equation of motion. You should get the same equation as the one obtained in (b.). Be sure to justify each step of your derivation.

(f.) Suppose we roughen the surface of the incline so that there's now a damping force on the block. The damping force acts parallel to the inclined plane, with force F = -b$\dot{s}$. What is the equation of motion now? Here, b is the *damping constant.*

(g.) Solve the damped simple harmonic motion equation you derived in (f.). Looking at your solution, state the condition under which the moving block comes to rest the fastest, without oscillating.

(h.) If the block is undergoing underdamped oscillation, what is the maximal heat energy that can be delivered to the ramp after time $\Delta t = \frac{2}{\gamma}$? Assume that the block starts out with an amplitude of A at $t = 0$. Recall that the amplitude decays over time in an underdamped oscillation.

(i.) How does the motion change if the gravitational constant is changed from g to $2g$? What changes, if any?

Problem 2. Coupled oscillators.

Two blocks, one with mass m and the other with mass M, are attached to each other through a Hookian spring with stiffness k. Write down the two equations of motion describing each block. Then solve these equations of motion to obtain the two normal modes. What is the angular frequency of each normal mode? What are the normal coordinates for each mode? Describe, using pictures and some words, what kind of motion each normal mode is describing. Then write down the most general solution to the system of two equations of motion. Describe physically what would happen to the two normal modes in the following three situations: (i.) $m < M$, (ii.) $m = M$, and (iii.) $m > M$. Also, describe what kind of motion would result if $m$ll$M$.

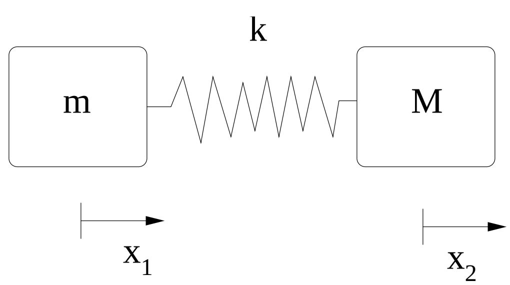

FIG. 2: Coupled oscillator. Notice that $m \neq M$. $x_1(t)$ and $x_2(t)$ describe the displacement of m and M with respect to their equilibrium positions respectively.

Problem 3. Coupled oscillation of lots of particles: Collective oscillation.

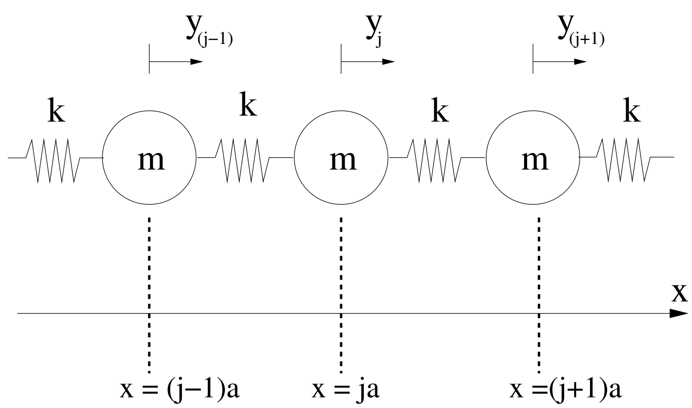

FIG. 3: One dimensional crystal.

Consider a chain of N atoms, with each atom of mass m bound to its adjacent neighbor through a spring with stiffness k. L is the total length of the chain, and in equilibrium, each atom is separated from its adjacent atom by distance a. $y_j(t)$ describes the *displacement* of the j-th atom from its equilibrium position. x describes the position of each particle relative to the left end of the wall. The left end of wall is $x = 0$ and the right end of the wall is $x = L$. The two ends of the chain is fastened to the two walls.

(a.) Derive the equation describing the collective oscillation of this chain of N particles when N is large. Be sure to justify each step of your derivation, specifying the *continuum limit* you take in the end. The equation describing this collective oscillation has a name, what is it?

(b.) Derive the total energy of the system, starting by writing down the energy of each spring and atom. Then summing all of them and taking the *continuum limit*, show that the total energy of the system is

$$E_{tot} = \int_0^L \frac{\rho}{2}(\frac{\partial y}{\partial t})^2 dx + \int_0^L \frac{D}{2}(\frac{\partial y}{\partial x})^2 dx \tag{1}$$

Problem 4. Solving a wave equation: Freely propagating wave.

(a.) Derive the most general solution to a one-dimensional wave equation. To remind you, the wave equation in one dimension is

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}, \tag{2}$$

where v is the wave speed. *(Hint: Think of a pulse moving to the right, and a pulse moving to the left with speed v).* Your answer should describe what *form* of $f(x,t)$ is a solution rather than specifying the exact expression of $f(x,t)$.

(b.) Write down the complex-number version of plane wave, moving to the right with speed v. Your answer should be in terms of wave number $k = \frac{2\pi}{\lambda}$, wave speed v, and complex number amplitude $\tilde{A}$. Show that this is a solution to the complex number version of the wave equation.

(c.) Write down the sinusoidal plane wave, moving to the right with speed v and wave length λ. As you did in (b.), your answer should be in terms of wave number k, the wave speed v, and amplitude A. Show that this is a solution to the real number version of the wave equation.

(d.) Next, write down the complex number representations of incident, reflected, and transmitted waves in the case of two strings (string A and string B) joined to each other at a junction (located at $x = 0$). The setup is shown in Figure 4.

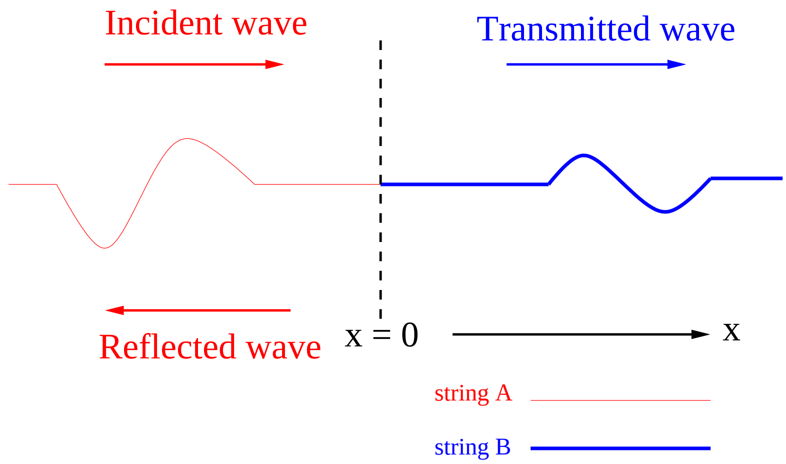

FIG. 4: . Incident and reflected waves travel in string A, while transmitted wave travels in string B. String A and B have mass density ρ_1 and ρ_2 respectively. Both are under constant tension T. In string A, wave travels at speed c_1 while in string B, wave travels at speed c_2.

(e.) Write down the expression for the (complex number representation) of the *resultant wave* you'd see with your eyes in string A, and in string B. Let $y_1(x,t)$ be the profile of the string A and $y_2(x,t)$ be the profile of string B.

(f.) Write down the two boundary conditions at the junction $x = 0$. For each boundary condition (represented by an equation) you write down, be sure to state one sentence describing what the equation *physically* represents.

Problem 5. Fourier series.

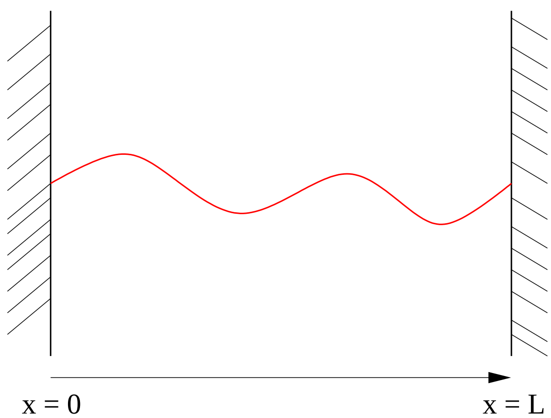

FIG. 5: Waves between two fixed walls.

(a.) By solving the wave equation obeyed by a string with its ends fixed at two walls separated by length L, show that the normal modes of the string are standing waves. For each normal mode describing a standing wave, determine what the wave number k, wave length λ, and the normal mode angular frequency ω are, Be sure to specify what the boundary conditions are when you're solving the wave equation.

(b.) What is the most general wave $y(x,t)$ that the string can support between the two walls? Express your answer in terms of the normal modes you found in (a.).

(c.) Suppose the string initially (at $t = 0$) has the shape described by $g(x)$:

$$g(x) = \frac{L}{4} sin(\frac{8\pi x}{2L}) \tag{3}$$

and that initially (t =0), the string is at rest. That is, no part of the string is moving because you're holding it with your hands. Immediately after $t = 0$, your hands release the string, after which it moves. What is $y(x, t)$ that describes the shape of the string at any time t afterwards? (*Hint*: If you're finding yourself solving complicated integrals, think again. While you can get the answer by doing the right integral, you can also get the answer much more easily by looking at $g(x)$ very carefully and thinking about the property of the integral $\frac{2}{L}\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx$.)

Problem 6. Electromagnetic (EM) waves

(a.) Starting from the Maxwell's equations, derive the wave equations obeyed by the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in free space (where there are no electric charges and currents). Also, show that the electric and magnetic fields making up the EM wave are perpendicular to each other. Recall that Maxwell's equations in free space are

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 \qquad\qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \qquad\qquad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \end{aligned} \tag{4}$$

where μ_0 and ϵ_0 are constants describing permeability of magnetic field and permittivity of electric field in vacuum. What is the speed of light c in terms of these two constants? What is it measured with respect to? The following vector calculus identity will help you out:

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \tag{5}$$

where $\vec{A}$ is any arbitrary vector.

(b.) Write down the plane wave describing the electromagnetic wave, propagating in the $+z$ direction with wave length λ at speed of light c. Your answer should consist of two equations, one for the electric field component of the EM wave, and the other for the magnetic field component of the EM wave. Be sure to define any new terms you introduce in your answer. You should be able to identify this from your resulting wave equations.

(c.) Why doesn't electromagnetic wave penetrate a good electric conductor? Why does it penetrate insulators (called dielectrics) so easily?

(d.) Describe how a wire-grid polarizer works. If you want to block out all the EM waves making up natural light (from sun, for example), how would you achieve this using two wire-grid polarizers? Recall that natural light consists of train of EM waves of all possible polarizations (randomly polarized).