MITES 2008 : Physics III : Oscillations & Waves

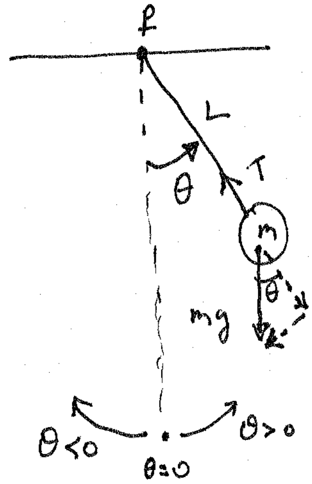
July 2, 2008

[Wed.]

pg1

Solution set 1

Problem 1



2 methods of deriving EOM

· Method ① : Using torque & angular momentum.

· Recalling that $\tau = \frac{dS}{dt}$ — $S \equiv$ Angular momentum, $\tau \equiv$ torque about the pivot P.

$\tau = -mg(\sin\theta)L$

(Tension & $mg\cos\theta$ component of gravity don't contribute to torque since both act parallel / anti-parallel to the string : line of moment.)

$\cdot\ S = I\dot{\theta} = mL^2\dot{\theta}$ — $I = mL^2$ ← moment of inertia. of mass m about the pivot point P.

$\therefore\quad -mgL\sin\theta = \frac{dS}{dt}$

$\Rightarrow\quad -mgL\sin\theta = mL^2\ddot{\theta}$ ← $\boxed{\text{Eq'n (1)}}$

But assuming small oscillations of pendulum, i.e. $|\theta(t)| \ll 1$ (in radians) at all t,

we have : $\sin\theta \approx \theta$ ← By Taylor approximation. (i.e. Assuming small amplitude θ_0)

$\therefore$ Eq'n (1) becomes:

$$-\cancel{m}gL\theta \approx \cancel{m}L^2\ddot{\theta} \Rightarrow \boxed{\ddot{\theta} + \frac{g}{L}\theta = 0}$$ ← EOM for Simple pendulum

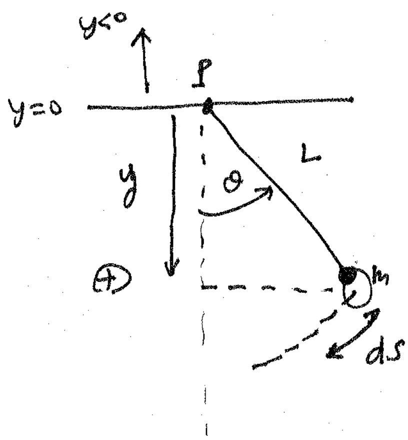
↳ we can identify $\omega_0^2 = g/L$

$\therefore$ $\boxed{\text{Angular frequency is } \omega_0 = \sqrt{g/L}.}$

Solving the EOM yields : $\boxed{\theta(t) = \theta_0 \cos\left(\sqrt{\tfrac{g}{L}}\,t - \phi\right)}$

$\boxed{\text{where } \theta_0 \equiv \text{Amplitude and } \phi \equiv \text{Phase shift are 2 free parameters}}$

· Method ②: Using Conservation of energy:



$y = L\cos\theta \quad > 0$ indeed. (thus my sign convention for y).

U_{grav} = potential energy of bob due to gravity.

$= -mgy$ ← ⊖ since as bob goes lower, ~~loss of energy due~~ potential energy should decrease. But since $y>0$ below ceiling, mgy would actually increase as y gets larger. So we need ⊖ in front to get $U_{grav} = -mgy$.

$= -mgL\cos\theta$

Kinetic energy:

$KE = \frac{m}{2}V^2$

To get V: ds = small (infinitesimal) arc length swept by bob.

$= l\,d\theta$

$\Rightarrow V = \frac{ds}{dt} = \frac{L\,d\theta}{dt} = L\dot{\theta}$

$\therefore \quad KE = \frac{m}{2}V^2 = \frac{mL^2\dot{\theta}^2}{2}$ ← "rotational KE" $\left(\frac{I\dot{\theta}^2}{2}\right)$

$E_{tot} = U_{grav} + KE = \frac{mL^2\dot{\theta}^2}{2} - mgL\cos\theta$

Conservation of energy:

$$0 = \frac{dE_{tot}}{dt} = \frac{mL^2}{\cancel{2}}\cancel{2}\dot{\theta}\underbrace{\frac{d\dot{\theta}}{dt}}_{\ddot{\theta}} + mgL(\sin\theta)\dot{\theta}$$

$$\Rightarrow \quad 0 = \dot{\theta}\left\{\cancel{m}L^2\ddot{\theta} + \cancel{m}gL\sin\theta\right\}$$

$$\Rightarrow \quad 0 = \ddot{\theta} + \frac{g}{L}\sin\theta \quad \leftarrow \boxed{\text{Eqn (2)}}$$

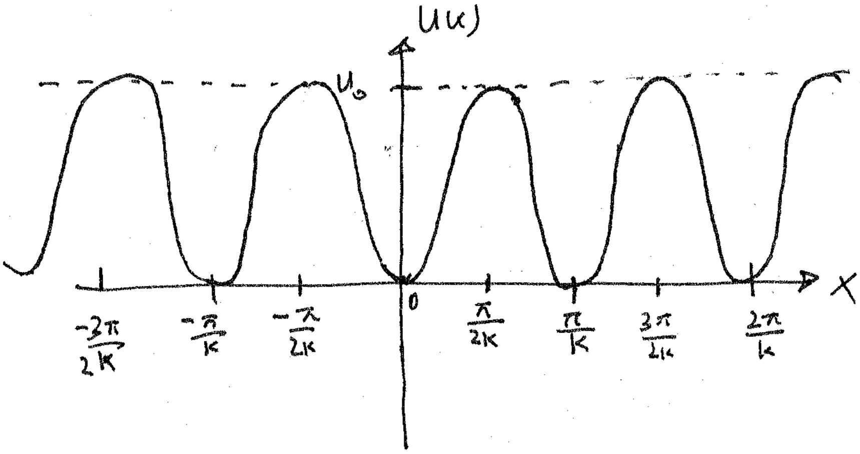
But assuming small oscillations (so that $|\theta(t)| \ll 1$ at all times):

$\sin\theta \approx \theta$ ← By Taylor approx.

$\therefore$ Eqn (2) becomes: $\boxed{0 = \ddot{\theta} + \frac{g}{L}\theta}$

Problem 2

1(a) $U(x) = U_0 \sin^2(kx)$



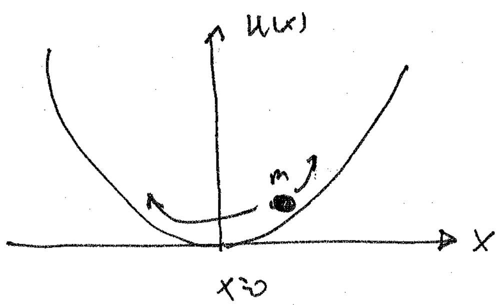
$k \equiv$ wave number of the standing wave intensity pattern.

(Note: wave # is defined as: $k = \frac{2\pi}{\lambda}$)

$\lambda \equiv$ wave length of standing wave.

$U_0 \equiv$ Maximum intensity of the optical trap.

1(b) Zooming into $x=0$ neighborhood in above diagram:



← looks like a quadratic well.

Using the hint: $\sin(z) \approx z$ (for $|z| \ll 1$),

we have: $\sin(kx) \approx kx$ (for $|kx| \ll 1$).

↑ "approx."

$\therefore \sin^2(kx) \approx (kx)^2 = k^2x^2$.

This is valid for the small oscillations we're interested in

(we're interested in oscillations with amplitude δx: where

$|k\,\delta x| \ll 1$

$\Rightarrow \boxed{|\delta x| \ll \frac{1}{k}.}$ (small oscillation approx.)

Again, use conservation of energy to derive EOM:

$$E_{tot} = \frac{m\dot{x}^2}{2} + U(x)$$

$$\approx \frac{m\dot{x}^2}{2} + \frac{k^2x^2}{2}U_0 \Rightarrow$$

conservation of energy:

$$\frac{dE_{tot}}{dt} = 0 = \frac{\cancel{2}m}{\cancel{2}}\dot{x}\ddot{x} + k^2 x \dot{x} U_0$$

$$\Rightarrow 0 = \dot{x}\left\{m\ddot{x} + k^2 x U_0\right\}$$

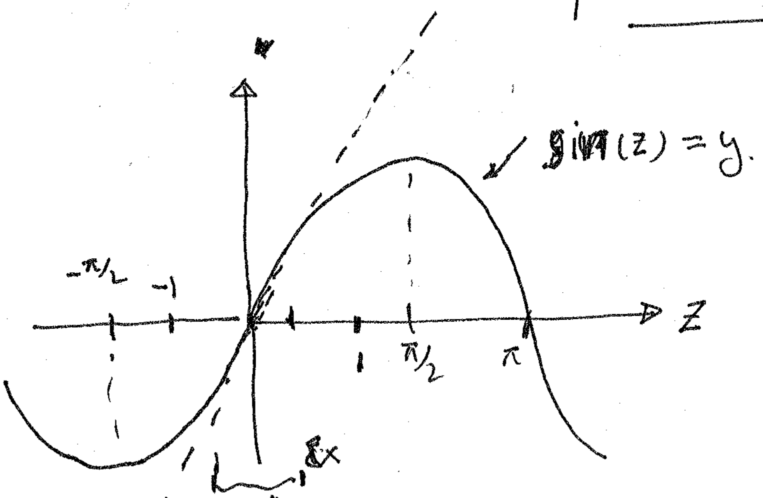
$$\Rightarrow \boxed{0 = \ddot{x} + \left(\frac{k^2U_0}{m}\right)x} \leftarrow \text{EOM}$$

From the EoM, we identify: $\omega_0^2 = \frac{k^2 U_0}{m}$

$$\Rightarrow \boxed{\omega_0 = \sqrt{\frac{k^2 U_0}{m}}} \leftarrow \text{Angular frequency.}$$

Solving the EoM yields: $\boxed{x(t) = \delta x \cos\left(k\sqrt{\frac{U_0}{m}}\, x - \phi\right)}$

$\delta x \equiv$ Amplitude of small oscillation



For small region ($|z| \ll 1$), $z \approx \sin(z)$. ← result of Taylor approx.

$y = z$.

Problem 3

(a) Recall that $m\frac{d^2x}{dt^2} + kx = 0$ was derived for SHO in class, starting from Newton's 2nd law in $F = ma$ format.

If, indeed, $F = ma$ was still true even when the mass of particle is changing, then indeed we'd get $m(t)\frac{d^2x}{dt^2} + kx = 0$ for this problem. But in fact, $F \neq ma$ when the mass is changing!

Recall that the real definition of force F is: $\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$ ← Newton's 2nd law.

where $\vec{p} \equiv$ momentum of particle.

And since $\vec{p} = m\vec{v} = m\dot{x}$,

$$F = \frac{d(m\dot{x})}{dt} = m\frac{d\dot{x}}{dt} + \dot{x}\frac{dm}{dt}$$

Hence, $$F = m\frac{d\dot{x}}{dt} + \dot{x}\frac{dm}{dt}$$
$$= m\ddot{x} + \underbrace{\dot{x}\frac{dm}{dt}}$$

Typically, this is zero in many problems we solve since m of particle usually constant. In that case, we get $F = m\ddot{x} = ma$.

But in our example, $\frac{dm}{dt} \neq 0$ ☒

[(b)] ∴ EOM of the "Sand-filling bucket-oscillator" is:

$$F = m\ddot{x} + \dot{x}\left(\frac{dm}{dt}\right)$$

$F = -kx$ ← ∵ The only force that acts on the block is spring force. (gravity & normal force cancel each other out)

$$\therefore \quad -kx = m\ddot{x} + \left(\frac{dm}{dt}\right)\dot{x}$$

$$\Rightarrow \boxed{0 = \ddot{x} + \underbrace{\left(\frac{(dm/dt)}{m}\right)}_{= 2\gamma}\dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{\omega_0^2}x} \quad \leftarrow \underline{\text{EOM}}.$$

$\omega_0 \equiv$ Natural ang. frequency

- $\frac{dm}{dt} > 0$ since sand is filling up the bucket.

So $\frac{(dm/dt)}{m} > 0$; just like damping coefficient 2γ.

- Looking at the EOM, this is indeed a damped oscillator with damping coefficient b, where:

$$\gamma = \frac{b}{2m} \quad ; \quad \text{but} \quad \gamma = \frac{(dm/dt)}{2m}$$

$$\therefore \quad \boxed{b = dm/dt} > 0 \leftarrow \underline{\text{damping constant}}.$$

☒

(c) Total energy of system : Since EOM is that of damped oscillator, the total energy $E_{tot}(t)$ should decrease over time. Let's see how this comes about by writing down $E_{tot}(t)$:

$$E_{tot}(t) = PE(t) + KE(t)$$

$$= \frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

Then,

(d)

$$\frac{dE_{tot}}{dt} = \cancel{\frac{2}{2}}kx\underbrace{\frac{dx}{dt}}_{=\dot{x}} + \cancel{\frac{2m}{2}}\dot{x}\ddot{x} + \frac{\dot{x}^2}{2}\left(\frac{dm}{dt}\right)$$

$$= \dot{x}\left\{kx + m\ddot{x} + \frac{(dm/dt)}{2}\dot{x}\right\}$$

$$= m\dot{x}\left\{\ddot{x} + \frac{(dm/dt)\dot{x}}{2m} + \omega_0^2 x\right\}$$ ← $\{\ldots\}$ Almost looks like EOM (Not exactly the EOM due to the "$\frac{1}{2m}$" instead of "$\frac{1}{m}$").

$$= m\dot{x}\left\{\underbrace{\left(\ddot{x} + \frac{(dm/dt)}{m}\dot{x} + \omega_0^2 x\right)}_{=0} - \frac{(dm/dt)}{2m}\dot{x}\right\}$$ ← EOM

$$= -\cancel{m}\dot{x}\frac{(dm/dt)}{2\cancel{m}}\dot{x}$$

$$= \boxed{-\frac{\dot{x}^2}{2}(dm/dt)} = \frac{dE_{tot}}{dt}.$$

$\underline{\dot{x}^2 \geq 0}$, $\underline{dm/dt > 0}$ $\therefore$ $\frac{dE_{tot}}{dt} = -\underbrace{(\ldots)}_{\geq 0} < 0.$

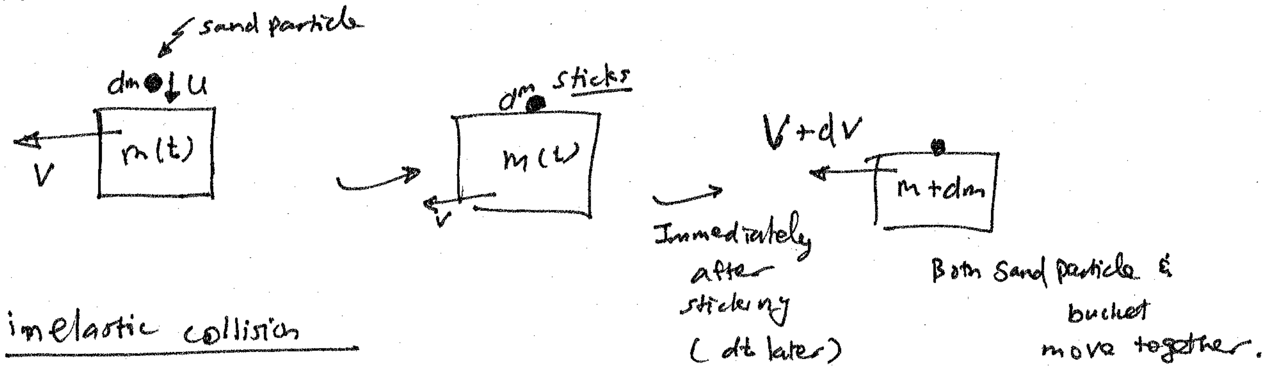
$\boxed{E_{tot} \text{ is } \underline{\text{decreasing}} \text{ over time}}$

- $E(t)$ decreasing over time. Where's this energy going?

Ans: Heat! energy is being dissipated.

But why is there heat?

- Think about what's happening to the sand particles as they land within the bucket.



This is an inelastic collision

↳ Sand particle sticks to a moving bucket immediately after landing within the bucket, and then immediately afterwards, the sand particle moves with the bucket at the same velocity as the bucket.

- In such inherently inelastic processes (i.e. sticking together, then immediately afterwards, acquiring new velocity) energy is always dissipated as heat.

$\boxed{1(e)}$ We have EOM: $0 = \ddot{x} + \frac{(dm/dt)}{m}\dot{x} + \omega_0^2 x \qquad \omega_0^2 \equiv k/m$

Now, If $m(t) = M + \beta t$: $dm/dt = \beta$, $\beta > 0$ since sand is filling up bucket.

$\therefore \quad 0 = \ddot{x} + \frac{\beta}{m}\dot{x} + \omega_0^2 x$

Let $2\gamma \equiv \beta/m$, then we have:

$0 = \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x$ ← familiar damped EOM.

- We know how to solve this from class. But let's solve it again to get practice.

over →

· To solve the EOM, we solve the $\mathbb{C}$-equivalent EOM: ($x \mapsto z$)
(since $\mathbb{C}$ is easier to do calculations w/)

$$0 = \ddot{z} + 2\gamma \dot{z} + z\omega_0^2 \leftarrow \text{EOM to solve.}$$

· Let $z(t) = Ae^{i\omega t}$, then plugging into EOM we get:

$$0 = -\omega^2 \cancel{z} + \omega_0^2 \cancel{z} + i\omega 2\gamma \cancel{z}$$

$$\Rightarrow \quad 0 = \omega^2 - 2i\gamma\omega - \omega_0^2 \leftarrow \text{quadratic eq'n in } \omega$$

& as of now, unknown quantity.

Solving for ω:

$$\omega_\pm = \frac{2\gamma i \pm \sqrt{-4\gamma^2 + 4\omega_0^2}}{2}$$

$$\Rightarrow \quad \omega_\pm = i\gamma \pm \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\equiv \text{ let } \tilde{\omega}}$$

$\therefore$ 2 values of ω obtained: ω_+ & ω_-.

So $z_1(t) = Ae^{i\omega_+ t}$ and $z_2(t) = Be^{i\omega_- t}$ are both sol'ns.

And so is $z(t) \equiv z_1(t) + z_2(t)$

$$= Ae^{i\omega_+ t} + Be^{i\omega_- t} \leftarrow \text{2 free parameters } (A \text{ \& } B)$$

$$= \boxed{e^{-\gamma t}\left[Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right]}$$

So this must be the general sol'n to EOM

Let's now take a look at various regimes:

→ over

Regime 1 : Underdamped : $\omega_0^2 > \gamma^2$. $\Rightarrow \tilde{\omega} \in \mathbb{R}$, > 0.

(This corresponds to $\frac{k}{m} > \frac{\beta^2}{4m^2} \Rightarrow 4mk > \beta^2 \Rightarrow \beta < 2\sqrt{km}$
$= 2\sqrt{\omega_0^2 m^2}$
$= 2\omega_0 m$.
($\beta < 2\omega_0 m$))

so, if rate of sand filling the bucket
(and sticking to the bucket, this being an inherently inelastic process)
is not too fast (i.e. $\beta < 2\omega_0 m$), the bucket + sand follows underdamped SHM.

· Its motion is described by $x(t)$ (NOT $Z(t)$: ← $\mathbb{C}$ #, so need to find the real part of this):

To find $x(t) = \mathrm{Re}(Z(t))$:

Note that $Z(t) = e^{-\gamma t}\left[Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right]$ $\quad A, B \in \mathbb{R}$

$$= e^{-\gamma t}\left[\left(A\cos(\tilde{\omega}t) \underbrace{-B}_{\equiv \tilde{B}}\sin(\tilde{\omega}t)\right) + i\left(A\sin(\tilde{\omega}t) \underbrace{-B}_{\equiv \tilde{B}}\sin(\tilde{\omega}t)\right)\right]$$

$$= e^{-\gamma t}\left[\underbrace{\left(A\cos(\tilde{\omega}t) + \tilde{B}\sin(\tilde{\omega}t)\right)}_{\mathrm{Re}} + i\underbrace{\left(A\sin(\tilde{\omega}t) + \tilde{B}\sin(\tilde{\omega}t)\right)}_{\mathrm{Im}}\right]$$

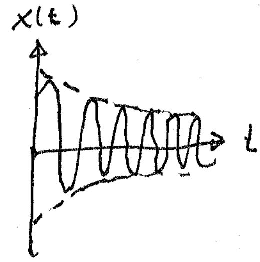
so:

$$x(t) = \mathrm{Re}(Z(t)) = e^{-\gamma t}\left[\underbrace{A\cos(\tilde{\omega}t) + \tilde{B}\sin(\tilde{\omega}t)}\right]$$

← From lecture, we know that we can rewrite this as $C\cos(\tilde{\omega}t - \phi)$ (more "physical" part)

$$\therefore \boxed{x(t) = e^{-\gamma t}C\cos(\tilde{\omega}t - \Phi)}$$

← position of bucket + sand when $\beta < 2\omega_0 m$ (underdamped).



Regime 2: Critical damping $\quad \omega_0^2 = \gamma^2 \Rightarrow \tilde{\omega} = 0.$

$$\Rightarrow \frac{k}{\cancel{m}} = \frac{\beta^2}{4m^{\cancel{2}}} \Rightarrow \beta = \sqrt{4mk}$$

$$= 2\sqrt{\omega_0^2 m^2}$$

When the rate of sand filling the bucket is exactly $2\omega_0 m$: $\Rightarrow \boxed{\beta = 2\omega_0 m}$

bucket + sand follows critically damped SHM.

Notice that when $\tilde{\omega} = 0$, $z(t)$ written on Pg 8 becomes: $z(t) = e^{-\gamma t}\underbrace{[A+B]}_{= C}$

$$= Ce^{-\gamma t}$$

Now, $z(t) = Ce^{-\gamma t}$ cannot be the most general sol'n since it has only one free parameter "c".

To find another sol'n, we try $z_2(t) \equiv e^{-\gamma t} t D$ ← free parameter.

$$= t z_1(t)$$

Check if z_2 satisfies EOM:

call $z_1(t) \equiv De^{-\gamma t}$

$\dot{z}_2 = z_1 + t\dot{z}_1$

$\ddot{z}_2 = 2\dot{z}_1 + t\ddot{z}_1$

plug in

$$\ddot{z}_2 + 2\gamma\dot{z}_2 + \omega_0^2 z_2$$

$$= 2\dot{z}_1 + t\ddot{z}_1 + 2\gamma\dot{z}_1 t + 2\gamma z_1 + \omega_0^2 t z_1$$

$$= t\underbrace{(\ddot{z}_1 + 2\gamma\dot{z}_1 + \omega_0^2 z_1)}_{= 0} + 2\dot{z}_1 + 2\gamma z_1$$

($\because z_1$ itself satisfies EOM)

$$= 2[\underbrace{-\gamma De^{-\gamma t} + \gamma De^{-\gamma t}}_{= 0}] = 0.$$

$\Rightarrow$ Indeed, z_2 is sol'n to EOM.

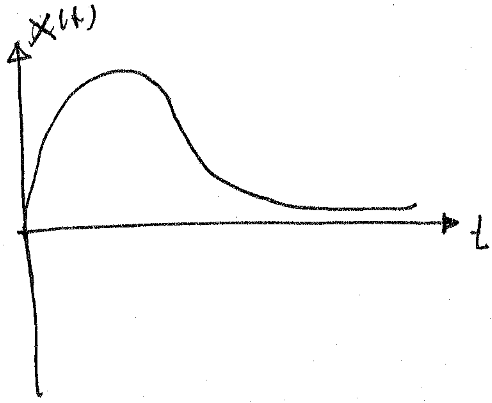
And since the EOM has "linearity" property (i.e. Our EOM is a linear differential eq'n),

$z(t) = z_1 + z_2$ is also a sol'n.

$= Ce^{-\gamma t} + tDe^{-\gamma t}$

$= e^{-\gamma t}[C + tD]$

And since it has 2 free parameters (C, D) it is the most general sol'n.



← Critically damped.

<u>Regime 3</u> : <u>Overdamped motion</u> : Occurs when $\omega_0^2 < \gamma^2$

$$\Rightarrow \frac{k}{\cancel{m}} < \frac{\beta^2}{4m^{\cancel{2}}} \Rightarrow \boxed{\beta > 2\omega_0 m}$$

• Occurs when the rate of sand filling the bucket β is larger than $2\omega_0 m$.

In this case, $\tilde{\omega} \equiv \sqrt{\omega_0^2 - \gamma^2}$ is imaginary so,

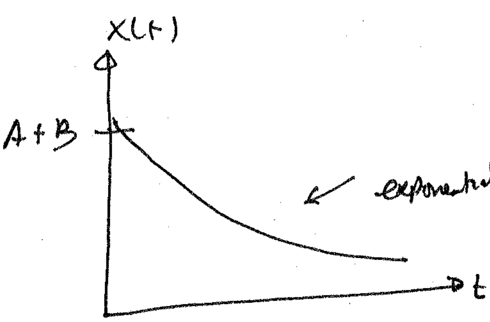
$= i\tilde{\tilde{\omega}}$ where $\tilde{\tilde{\omega}} = \sqrt{\gamma^2 - \omega_0^2} \in \mathbb{R}$ > 0.

So, <u>Z(t) on pg 8 becomes</u> :

$$Z(t) = e^{-\gamma t}\left[A e^{-\tilde{\tilde{\omega}} t} + B e^{\tilde{\tilde{\omega}} t}\right]$$

$$\boxed{X(t) = A\exp\left[-(\gamma+\tilde{\tilde{\omega}})t\right] + B\exp\left[-(\gamma-\tilde{\tilde{\omega}})t\right]}$$

↑ Already real, so call it "x(t)".



↙ exponential decay.

Since $\gamma + \tilde{\tilde{\omega}} > \gamma - \tilde{\tilde{\omega}}$,

$e^{-(\gamma+\tilde{\tilde{\omega}})t}$ decays to zero first.

Hence, at large t, $e^{-(\gamma-\tilde{\tilde{\omega}})t}$ is the <u>dominant term</u> ($\because$ it's the slowly dying term.)

Problem 4

(a) **(i)** $M \equiv$ total mass of Earth.

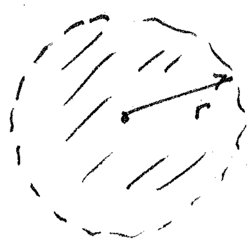
$R \equiv$ radius of Earth.

In this problem, we assume that the Earth has uniform ~~mass~~ density.

mass density (per volume)

So $$\rho \equiv \frac{M}{\text{Volume of Earth}} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Hence: within sphere of radius r embedded within Earth:



$$M(r) = \rho \, \tfrac{4}{3}\pi r^3$$
$$= \frac{\cancel{3}M}{\cancel{4}\pi R^3}\,\frac{\cancel{4}}{\cancel{3}}\pi r^3 = \boxed{M\left(\frac{r}{R}\right)^3}$$

$\therefore$ Mass enclosed within the dashed sphere of radius r is $\boxed{m(r) = M\left(r/R\right)^3}$

(ii)

- Gravity points towards center of the dashed sphere as shown. (i.e. ~~radially~~ radially inwards.)

So, by Newton's law of gravity:

$$F_{grav.}(r) = \frac{G\,M(r)\,m}{r^2}$$

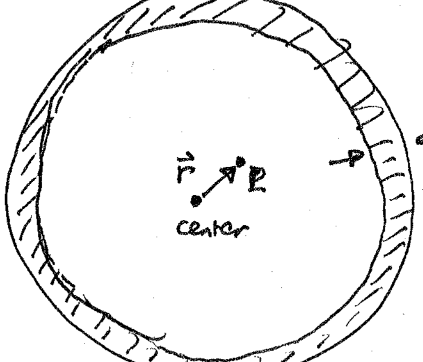
$m \equiv$ mass of person standing at position $\vec{r}$. (i.e. on surface of dashed sphere.)

$$= \frac{G\,M\left(\frac{r}{R}\right)^3 m}{r^2}$$

$$= \boxed{\frac{GMm}{R^3}\,r}$$ points (radially towards center)

<u>Sanity check</u>: When $r = R$, we get the familiar $F_{grav}(R) = \dfrac{GMm}{R^2}$.

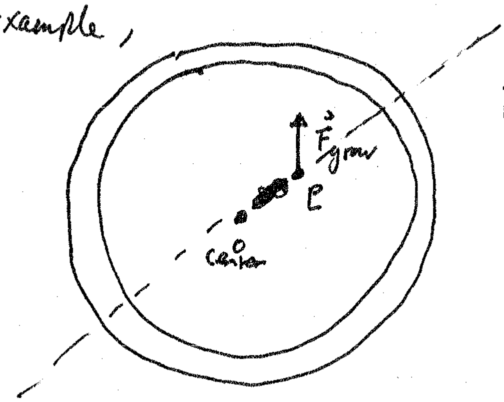
(iii)



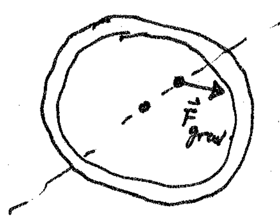
dr = infinitesimal thickness of ~~slab~~ spherical slab.

By rotational symmetry of sphere (i.e. sphere looks the same from all directions), the gravitational force at point P, due to the slab of mass must point ~~—~~ radially toward or away from the center of sphere. (If the force is non-zero.)

For example,



$\vec{F}_{grav.}$ cannot be pointing in the direction shown in this diagram since your friend standing behind this page and looking at the same sphere as you would see:

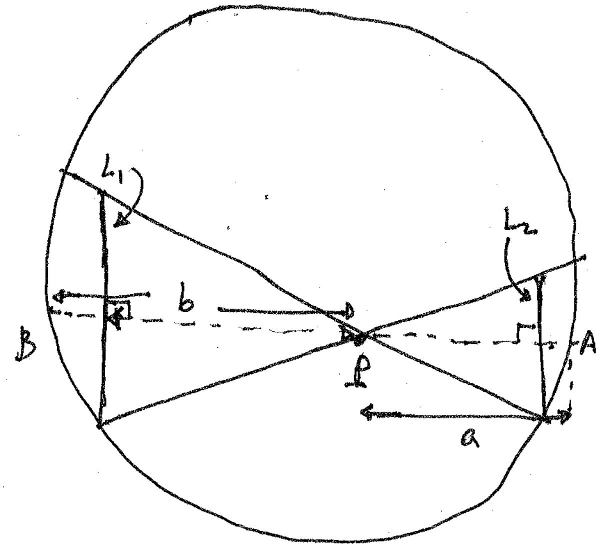


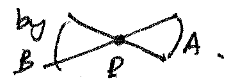
But since sphere looks the same in all directions, this cannot be!

Why should gravity prefer left hand side of dashed line shown, instead of right hand side?

. Usage of symmetry such as this is important in physics. ∎

$\boxed{\text{(iv)}}$



Let B & A be the ends of ~~some~~ sphere that ~~the~~ are spanned by  ..

~~Let~~ b be distance from P to piece B and a be distance from P to piece A, ~~the~~ As shown, draw 2 perpendicular bases of 2 cones ~~of~~ (L_1 and L_2). Call these bases B' & A' respectively. The ratio of the areas of A' and B' is then a^2/b^2.

The key here is to note that the angle between the planes of A and A' is the same as that between B and B'.

This is because the chord between A & B meets the circle at equal angles at its ends.

Hence, the ratio of the areas of A & B is also a^2/b^2.

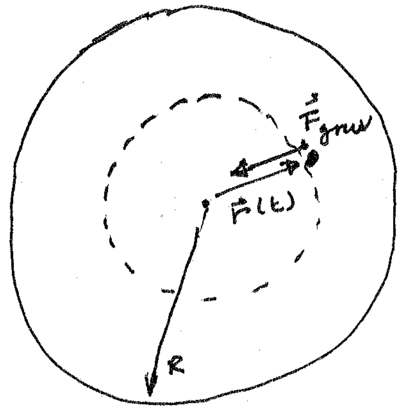
But gravitational force decrease ~~as~~ as $\frac{1}{r^2}$.

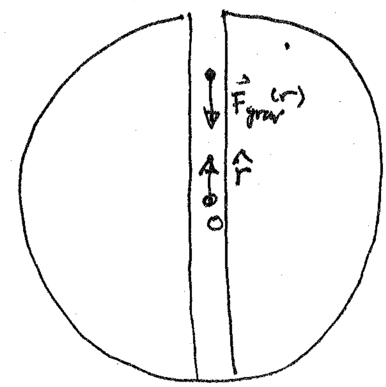
$\therefore$ Forces at P due to A & B are equal in magnitude but opposite in direction. Thus, net force at P $= 0$.

Since our Earth can be thought of as being made up by a series of concentric spherical shells, this proves that the net force at position $\vec{r}$ on Fig 1 on Part 1 handout must be due to only the mass enclosed within the dashed sphere.

$\blacksquare$

(v) Since only the mass enclosed within the ~~dashed~~ sphere exerts net force on the person at position $\vec{r}$





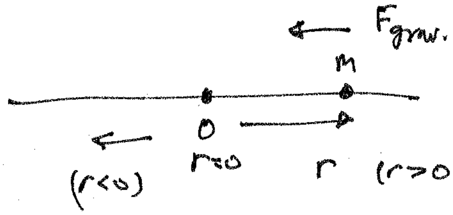
$$\vec{F}_{grav}(r) = -\frac{GMm}{R^3} r \hat{r}$$

$\hat{r} \equiv$ unit vector (i.e. $|\hat{r}| = 1$) pointing radially outwards from the center of Earth.

Or, if you prefer not to think about radial vectors, (this comes up when dealing w/ polar coordinates. You'll learn in calculus.)

note that this is actually a 1D problem:

Note that $r=0$ (center of Earth) is equilibrium position since $F_{grav}(r=0) = 0$.



$\therefore$ EOM is:

$m \equiv$ mass of person

$$\cancel{m}\ddot{r} = -\left(\frac{GM\cancel{m}}{R^3}\right) r$$

$$\Rightarrow \boxed{\ddot{r} + \left(\frac{GM}{R^3}\right) r = 0} \leftarrow \text{EOM of SHO!}$$

Constant $\equiv \omega_0^2$

$\therefore$ Indeed, the person would oscillate simple harmonically back & forth between the 2 poles in the tunnel, with angular frequency $\boxed{\omega_0 = \sqrt{\frac{GM}{R^3}}}$

$\Rightarrow$ w/ frequency f: $f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{GM}{R^3}}$

and $\therefore$ period $\boxed{T = \frac{1}{f} = 2\pi\sqrt{R^3/GM}}$.

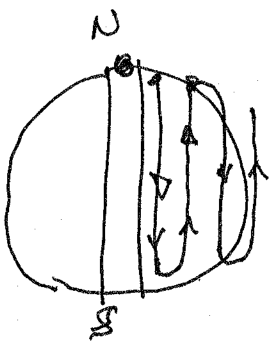
over

Since $\omega_0 = \sqrt{\frac{GM}{R^3}}$

- If mass of Earth M increases: ω_0 increases like $\sim \sqrt{M}$.
  (so person oscillates ~~at rea~~ between 2 poles faster.)
  (~~[illegible]~~ shorter period)
- If mass of person m changes: no effect on ω_0.
- If radius of Earth R increase: ω_0 decreases like $\sim \frac{1}{R^{3/2}}$.
  $\Rightarrow$ person oscillates between the 2 poles slower
  (longer period.)

$\boxed{(b)}$ Period $T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{R^3}{GM}}$.

But,



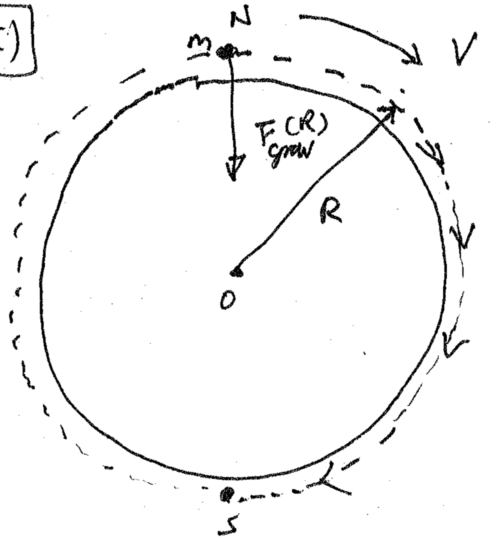
← shows the path of ~~person to~~ d person in tunnel.

~~So Time to~~

From this diagram, time taken by the person to travel from N to S is $\frac{T}{2}$.

$\Rightarrow$ $$\boxed{t_{tunnel} = \pi \sqrt{\frac{R^3}{GM}}}$$ ← time taken to go from N to S pole through tunnel.

(c)



<u>centripetal motion</u>

orbit speed $= V$

$$\Rightarrow \frac{mV^2}{R} = F_{grav.}(R) = \frac{GMm}{R^2}$$

$$\Rightarrow \boxed{V = \sqrt{\frac{GM}{R}}} \leftarrow \text{orbital speed.}$$

- If Earth's radius increases, V decreases like $\sim \frac{1}{\sqrt{R}}$ (assuming M doesn't change.)

(d) Distance from N to S pole in the circular orbit case is ½ circumference of circle.

$$\Rightarrow \quad \text{distance} = \frac{2\pi R}{2} = \pi R.$$

$\therefore$ <u>time taken to get from N to S pole using circular orbit</u>:

$$t_{orbit} = \frac{\pi R}{V} = \pi R \sqrt{\frac{R}{GM}} = \boxed{\pi \sqrt{\frac{R^3}{GM}}}$$

(e) Hence, $\boxed{t_{orbit} = t_{tunnel}}$!

$\therefore$ Both modes of travel take the same time.

- To see why the tunnel is so dangerous, consider (and calculate) the maximum speed reached in the tunnel. Max. speed is obtained when the person reaches the center of Earth.

$r(t) = R\cos(\omega_0 t)$ ← position of person (sol'n of SHM).

so, $\dot{r}(t) = -\underset{\sim}{R}\omega_0 \sin(\omega_0 t)$

↑ Max. speed. (obtained when $t = \frac{\pi}{2\omega_0}$ ← when person at center of Earth.

$$\dot{r}_{max} = R\omega_0 = R\sqrt{\frac{GM}{R^3}} = \sqrt{\frac{GM}{R}} \approx \sqrt{\frac{10^{24}\cdot 10^{-11}}{10^3}}\ m/s \approx 10^5\ m/s$$

↑ plug in the actual #'s (order of magnitudes would do.)

$M \approx 10^{24}$ kg.
$G \approx 10^{-11}\ m^3\ kg^{-1} s^{-2}$
$R \approx 10^3$ km $= 10^6$ m.

Order of magnitude estimation gives: $v_{max} = \dot{r}_{max} \simeq \boxed{100\ km/sec.}$ (!!)

↑ dangerous speed to be traveling within tunnel

In fact, $10^5\ m/s \sim \frac{1}{1000} \times$ speed of light. !

(speed of light $c = 3\times10^8\ m/s$).

∎

Problem 5

(a) Want to find the equilibrium position of the 2 blocks.

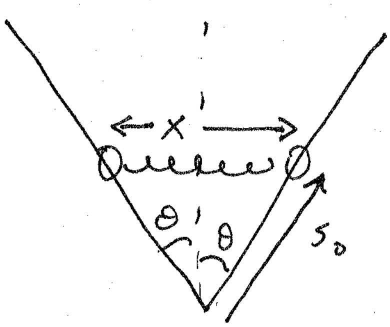
Let $x_0 \equiv$ rest length of spring.

Intuition tells us that in equilibrium, spring is compressed.

Let $\quad x_0 - x \equiv \Delta x > 0$

x: length of spring in equilibrium.

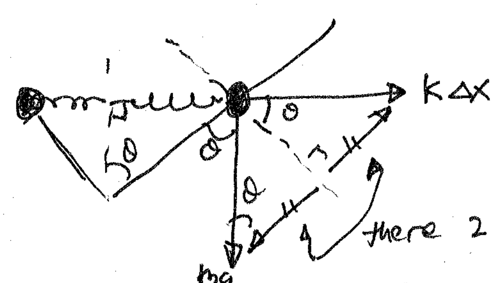
$\therefore$ compressed from ~~equil. position~~ rest length.



Goal: Find s_0 in terms of x_0

- since motion only along rail, break all forces into component $\parallel$ and $\perp$ to rail, then only think about $\parallel$ components (parallel to rail)

so:



these 2 cancel out in equilibrium.

$\therefore \quad k(\Delta x)\sin\theta = mg\cos\theta$

$\Rightarrow \quad \Delta x = \frac{mg}{k}\cot\theta$

$\Rightarrow \quad x_0 - x = \frac{mg}{k}\cot\theta$

$\Rightarrow \quad x = x_0 - \frac{mg}{k}\cot\theta \quad$; but $\quad x = 2 s_0 \sin\theta$

$2 s_0 \sin\theta = x_0 - \frac{mg}{k}\cot\theta$

$\Rightarrow \quad \boxed{s_0 = \frac{1}{2\sin\theta}\left[x_0 - \frac{mg}{k}\cot\theta\right]}$

distance from vertex of ~~equil~~ rail to mass when in equilibrium.

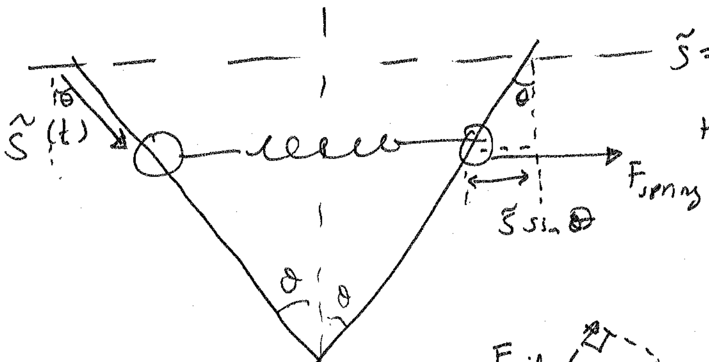
1(b) Let $\tilde{s}(t) \equiv s(t) - s_0$ ← change of ~~var~~ variable.

$\tilde{s}(t)$: displacement from equilibrium s_0

$s(t)$: position of m from vertex of wedge.

· As we found in class, (lesson on forced oscillation (w/ constant force) (Here, $mg\cos\theta$)) we can ignore $mg\cos\theta$ acting on the bead as long as we're only concerned w/ displacement $\tilde{s}(t)$ from equilibrium. (see lecture note #3 if ~~this~~ this is not clear.)

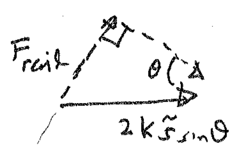
Thus



Here, spring compressed by $2\tilde{s}\sin\theta$.

$\Rightarrow F_{spring} = 2k\tilde{s}\sin\theta$.

"2" since the other mass gets compressed too.



So; ~~force~~ → component of spring force parallel to rail is:

$$F_{rail} = -2k\tilde{s}\sin^2\theta$$

⊖ since opposite the direction of $\tilde{s}$.

Hence: EOM:

$$m\ddot{\tilde{s}} = -2k\tilde{s}\sin^2\theta$$

$$\Rightarrow \boxed{\ddot{\tilde{s}} + \left(\frac{2k\sin^2\theta}{m}\right)\tilde{s} = 0}$$

↑ EOM is that of SHO.

$$\omega_0^2 = \frac{2k\sin^2\theta}{m} \Rightarrow f = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{2k\sin^2\theta}}$$

$$= \boxed{\frac{\pi}{\sin\theta}\sqrt{\frac{2m}{k}}}$$

↑ frequency of oscillation.

Solving above EOM: we get:

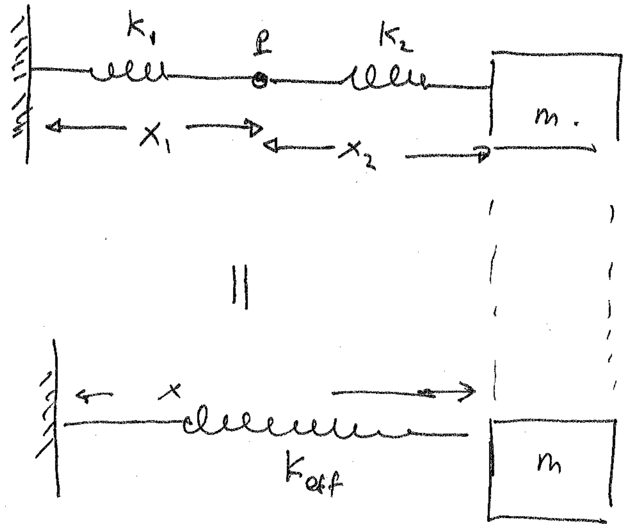
$$\tilde{s}(t) = \tilde{s}_0\cos(\omega_0 t + \phi) = \boxed{\tilde{s}_0\cos\left(\sqrt{\frac{2k}{m}\sin^2\theta}\, t + \phi\right)}$$

displacement from equilibrium

~~P~~

Problem 6

(a)



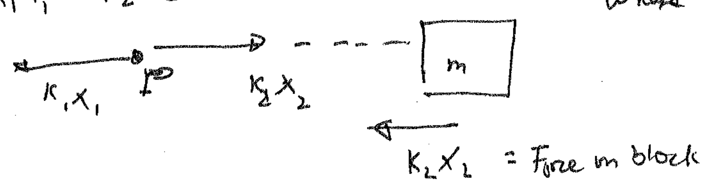
Without loss of generality, we assume that the rest length of spring 1 & spring 2 are both zero.
(Note: you don't have to assume this; you can instead say Z_1 and Z_2 are the rest lengths of 2 springs respectively, and you'll still get the same answer in the end. Try it!)

- $k_{eff} X$ = force acting on block m. Want to know what k_{eff} is.

Consider point P in the diagram above: Joining spring 1 (stretched by x_1) and spring 2 (stretched by x_2).
Since P is massless, we need $F_{net} = 0$ there.

so:

$$k_1 x_1 = k_2 x_2 \qquad \text{where } x_1 + x_2 = X$$



So:

$$x_1 = \frac{k_2 x_2}{k_1}$$

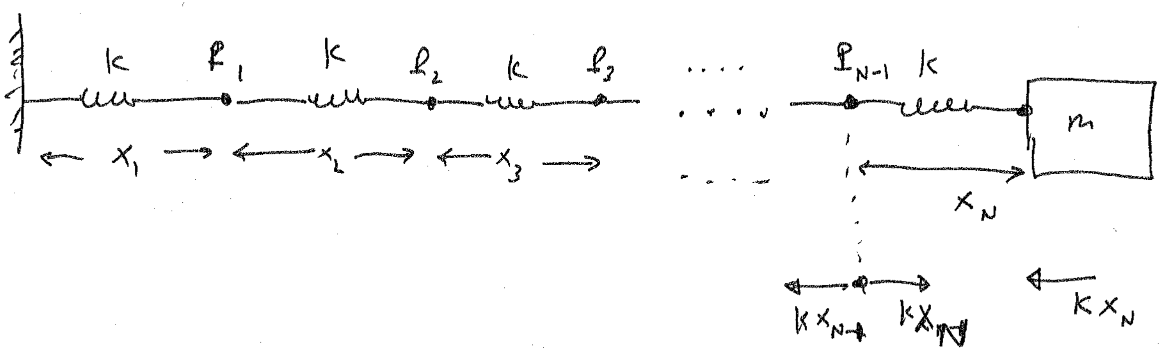
$\therefore$

$$k_{eff} X = \text{force on block } m = k_2 x_2$$

$$\Rightarrow k_{eff}\left(\frac{k_2 x_2}{k_1} + x_2\right) = k_2 x_2$$

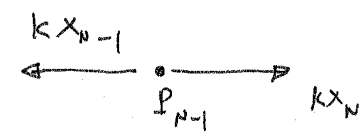
$$\Rightarrow \boxed{k_{eff} = \frac{k_1 k_2}{k_1 + k_2}}$$

(b) N identical springs, each w/ spring constant k.



Again, on the block, force $= kx_N$.

At massless point P_{N-1}, $F_{net} = 0$. $\Rightarrow$



$\Rightarrow$ $x_{N-1} = x_N$

In fact, you can see that for any j:

(At P_j): $kx_j = kx_{j+1} \Rightarrow x_j = x_{j+1}$.

Thus:

$$X = \sum_{j=1}^{N} x_j = N x_N.$$

And

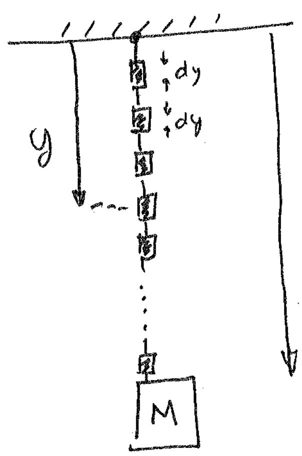
$$k_{eff} X = \text{Force on block } m = kx_N$$

$$\Rightarrow k_{eff} N x_N = kx_N$$

$$\Rightarrow \boxed{k_{eff} = \frac{k}{N}}$$ ← effective spring constant.

Problem 7

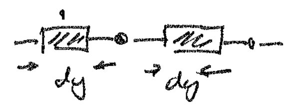
(a) Consider our argument on pg 22 -
We can think of the spring to be made up of N identical springs attached to one another (end-to-end) in series. (where N is very large, and each spring has infinitesimal length dy).



Assuming uniform mass density (mass / length),

$$\lambda \equiv \frac{m}{L} \quad \leftarrow \text{linear mass density.}$$

$$\Rightarrow \quad dm = \lambda \, dy$$



dm: infinitesimal mass of the infinitesimal element of spring.

If the block at $y = L$ moves by distance y_0, then the mass element at $y = y_1$ must move by $\left(\frac{y_1}{L}\right) y_0$.

(e.g. so, if $y_1 = 0$, (at the ceiling), then it doesn't move at all: $\frac{y_1}{L} y_0 = 0$.
If $y_1 = L$, then $\left(\frac{L}{L}\right) y_0 = y_0$.)

∴ The mass element at y must move w/ speed $V_{\text{mass element}} = \frac{y\,V}{L}$, where V = speed of block at bottom.

∴ KE of infinitesimal mass element of spring is:

$$KE = \frac{1}{2}(dm)\, V_{\text{mass element}}^2$$

$$= \frac{1}{2}(\lambda\, dy)\left(\frac{y}{L}V\right)^2$$

$$= \boxed{\frac{1}{2}\frac{m}{L}\, dy \left(\frac{yV}{L}\right)^2}$$

☐

(b)

$$E_{tot} = KE_{spring} + KE_{block} + \underbrace{\frac{ky^2}{2}}_{\text{PE of spring.}}$$

$$KE_{spring} = \sum \text{KE of each infinitesimal mass element}$$

$$= \int_0^L \frac{1}{2}\left(\frac{m}{L}\,dy\right)\left(\frac{yv}{L}\right)^2 = \frac{m}{2L}\frac{v^2}{L^2}\int_0^L y^2\,dy$$

$$= \frac{mv^2}{2L^3}\left.\frac{y^3}{3}\right|_0^L$$

$$= \frac{mv^2}{6L^3}L^3 = \frac{mv^2}{6}$$

$$\therefore \quad E_{tot} = \frac{mv^2}{6} + \frac{Mv^2}{2} + \frac{ky^2}{2} \qquad v = \dot{y}$$

$$= \left(\frac{m}{6} + \frac{M}{2}\right)\dot{y}^2 + \frac{ky^2}{2}.$$

Energy conservation:

$$\frac{dE_{tot}}{dt} = 0 = \left(\frac{m}{6} + \frac{M}{2}\right)2\dot{y}\ddot{y} + ky\dot{y}$$

$$\Rightarrow \quad 0 = \left[\left(\frac{m}{3} + M\right)\ddot{y} + ky\right]\dot{y}$$

$$\Rightarrow \quad \boxed{0 = \ddot{y} + \underbrace{\frac{k}{(M + m/3)}}_{\omega^2 = \frac{k}{M + m/3}}y} \quad \leftarrow \text{EOM of SHO}$$

Hence:

$$\boxed{\omega = \sqrt{\frac{k}{M + m/3}}}$$

$\boxed{E}$

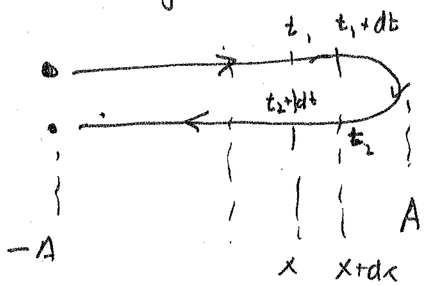
Problem 8

$X(t) = A\cos(\omega t + \phi)$. ← describes SHO position.

- Say it takes time dt to go from x to $x+dx$ (infinitesimal displacement).

- The probability of finding the oscillating particle between position x and $x+dx$ is equal to the probability that you'll be watching the particle between time t and $t+dt$ where the particle is at x at time t, and is at $x+dx$ at time $dt+t$.

But since the particle is simple harmonically oscillating, the particle actually passes through the interval $[x, x+dx]$ <u>twice</u> in one cycle.



(Once at t_1 and the other at t_2).

So: Probability of finding particle within $[x, x+dx]$

$$= \frac{2\,dt}{T}$$

$T \equiv$ period of SHM

$\omega T = 2\pi$

$$= \frac{2\,dt}{2\pi/\omega}$$

$$= \frac{\omega\,dt}{\pi} \quad \leftarrow \text{eqn (1)}$$

Now, $dx = \left|\frac{dx}{dt}\right| dt$

$\left|\frac{dx}{dt}\right| = |\dot{x}|$ ← Absolute value since I want the speed (Not velocity)

$$\Rightarrow \quad dt = \frac{dx}{|\dot{x}|} = \frac{dx}{+\omega A\sin(\omega t + \phi)}$$

So, eqn (1) becomes:

$$\left(\text{Probability of finding particle within } [x, x+dx]\right) = \frac{\omega\,dt}{\pi} = \frac{\cancel{\omega}\,dx}{\pi\,\cancel{\omega} A\sin(\omega t+\phi)}$$

But, $A\sin(\omega t + \phi) = \sqrt{A^2 - A^2\cos^2(\omega t + \phi)}$

$$= \boxed{\frac{dx}{\pi\sqrt{A^2 - x^2}}}$$

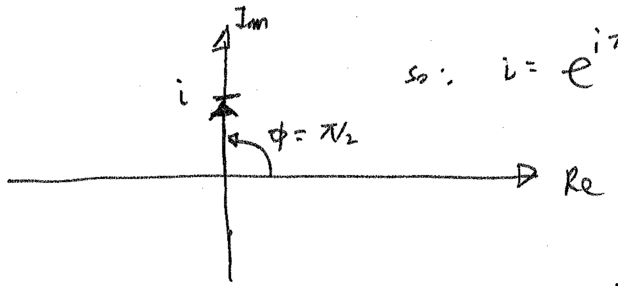
Problem 9 : See lecture notes on damped SHM.

(or, look at pgs 9, 10, 11 of this solution set.).

Problem 10

$$i^{i} = ?$$

First, in $\mathbb{C}$-plane we studied in class:



so: $i = e^{i\pi/2} = \underbrace{\cos(\pi/2)}_{=0} + i\underbrace{\sin(\pi/2)}_{=1}$ indeed.

Hence, $$i^{i} = \left(e^{i\pi/2}\right)^{i} = e^{-\pi/2} \approx 0.2079 < 1.$$

$\boxed{\text{Hence, you should keep your \$1 instead of accepting } i^{i}\ \$.}$