MITES 08 : Physics III

Solution set 2

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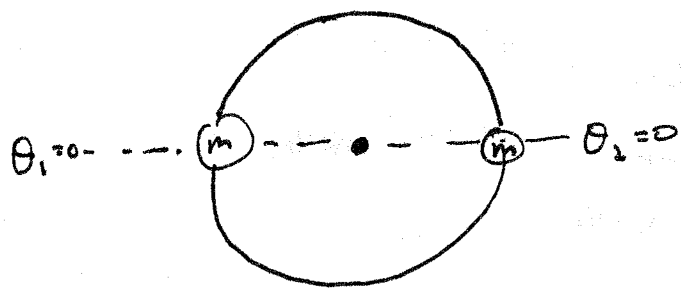
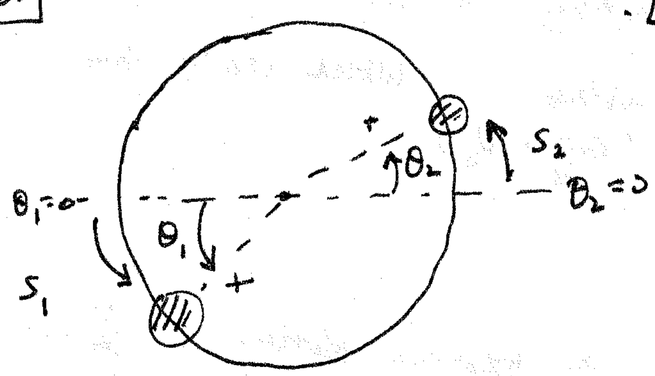
Problem 1 Ring Oscillator

1(a) In equilibrium: $\theta_1 = 0$, $\theta_2 = 0$



No net force on either bead.

1(b)



· Let s be arc length

So $s_1(t) = R\,\theta_1(t)$

$s_2(t) = R\,\theta_2(t)$

(we're measuring the arclength between a bead at its present position $\theta(t)$ and its equilibrium position $\theta = 0$.)

Then Newton's 2nd law, applied to motion along the circle becomes:

$$m\ddot{s}_1 = -k(s_1 - s_2) + k(s_2 - s_1) \quad \ldots \text{EOM } ①$$

$$m\ddot{s}_2 = k(s_1 - s_2) - k(s_2 - s_1) \quad \ldots \text{EOM } ②$$

But $s_j = R\theta_j$ so:

$$mR\ddot{\theta}_1 = -kR(\theta_1 - \theta_2) + kR(\theta_2 - \theta_1) \quad \ldots \text{EOM } ①$$

$$mR\ddot{\theta}_2 = +kR(\theta_1 - \theta_2) - kR(\theta_2 - \theta_1) \quad \ldots \text{EOM } ②$$

$\omega_0 = \sqrt{\frac{k}{m}}$

Rearranging:

$$\boxed{\begin{aligned} \ddot{\theta}_1 &= -2\omega_0^2\theta_1 + 2\omega_0^2\theta_2 \quad \ldots \text{EOM } ① \\ \ddot{\theta}_2 &= 2\omega_0^2\theta_1 - 2\omega_0^2\theta_2 \quad \ldots \text{EOM } ② \end{aligned}}$$

$\boxed{\ell \text{ (d)}}$

$\boxed{\text{(c)}}$ From (b), the EOMs are:

$$\begin{cases} \ddot{\theta}_1 + 2\omega_0^2\theta_1 - 2\omega_0^2\theta_2 = 0 & \cdots ① \\ \ddot{\theta}_2 - 2\omega_0^2\theta_1 + 2\omega_0^2\theta_2 = 0 & \cdots ② \end{cases}$$

1 : Adding EOMs ① & ② together:

$$(\ddot{\theta}_1 + \ddot{\theta}_2) = 0. \quad \text{Let} \quad \Theta \equiv \theta_1 + \theta_2.$$

← "Big theta"

$$\Rightarrow \boxed{\ddot{\Theta} = 0}$$

← "Effective EOM" is above equation.

Solution of this EOM is found by noticing that the acceleration $\ddot{\Theta}$ is zero. ⇒ constant velocity motion (call it V_0) which starts from some initial position Θ_0 at $t=0$.

$$\Rightarrow \boxed{\Theta(t) = V_0 t + \Theta_0}$$

← General solution to $\ddot{\Theta} = 0$
(V_0, Θ_0 : free parameters)

You can check that this is indeed solution to $\ddot{\Theta}=0$ by plugging it back into the eqn $\ddot{\Theta}=0$.

It's also the general solution to $\ddot{\Theta}=0$ ← 2nd order differential eqn since it has 2 free parameters V_0 and Θ_0.

Next,

2: Subtracting EOMs (1) & (2) on (Pg 2):

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + 4\omega_0^2\theta_1 - 4\omega_0^2\theta_2 = 0$$

$$\Rightarrow \boxed{(\ddot{\theta}_1 - \ddot{\theta}_2) + 4\omega_0^2(\theta_1 - \theta_2) = 0}$$

Let. $\Omega \equiv \theta_1 - \theta_2$.

then: ↳ "Big omega"

$$\boxed{\ddot{\Omega} + 4\omega_0^2\,\Omega = 0}$$

↳ "Effective EOM" 2

← But this is just the EOM of a simple harmonic oscillator with angular frequency $\tilde{\omega}$:

$$\tilde{\omega} = \sqrt{4\omega_0^2} = 2\omega_0$$

- we can thus immediately write down the general solution to above EOM;

$$\boxed{\Omega(t) = C\cos(2\omega_0 t - \phi).}$$

← General solution to the effective EOM #2.

(C, ϕ : free parameters)

Now, using the solutions to the 2 effective EOMs, we can obtain general solutions describing $\theta_1(t)$ and $\theta_2(t)$:

since $\Omega = \theta_1 - \theta_2$

$\Theta = \theta_1 + \theta_2$

we have:

$$\boxed{\begin{aligned} \theta_1(t) &= \frac{\Omega(t) + \Theta(t)}{2} \\ \theta_2(t) &= \frac{-\Omega(t) + \Theta(t)}{2} \end{aligned}}$$

Thus,

$$\theta_1(t) = \frac{C}{2}\cos(2\omega_0 t - \phi) + \frac{V_0}{2}t + \frac{\Theta_0}{2}$$

$$\theta_2(t) = -\frac{C}{2}\cos(2\omega_0 t - \phi) + \frac{V_0 t}{2} + \frac{\Theta_0}{2}$$

But ~~to~~ to avoid keep writing $1/2$, we define

$$\tilde{C} \equiv C/2, \quad \tilde{V}_0 = V_0/2, \quad \tilde{\Theta}_0 = \Theta_0/2$$

And writing in vector form:

$$\Rightarrow \boxed{\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = (\tilde{V}_0 t + \tilde{\Theta}_0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{C}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega_0 t - \phi)}$$

$\tilde{V}_0$, $\tilde{\Theta}_0$, $\tilde{C}$, ϕ are 4 free parameters.

& 2 free parameter for each particle $\Downarrow$ suitable # of parameters

There are 2 normal modes. ~~(when added together, giving)~~

- Normal mode #1 : $\omega_1 = 0$ ← Normal mode angular frequency!

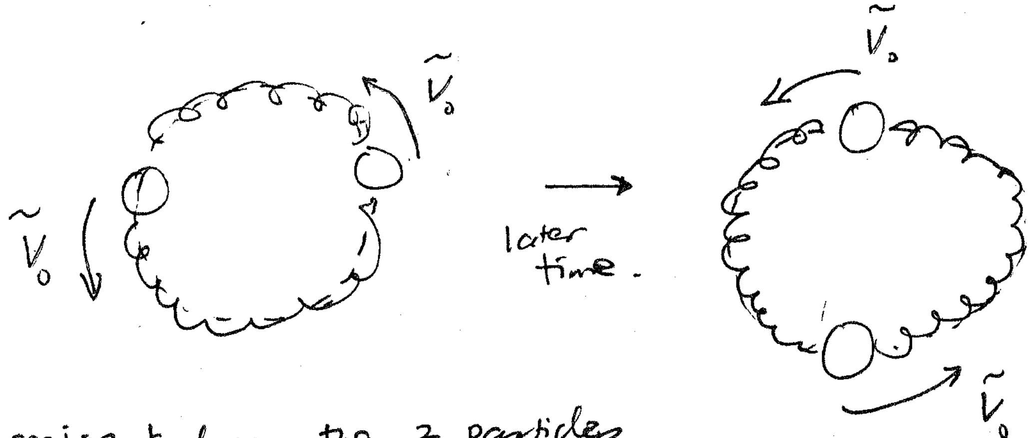
$$\boxed{(\tilde{V}_0 t + \tilde{\Theta}_0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv V_1}$$

↑ "defined as"

- Normal mode #2 : $\omega_2 = 2\omega_0$ ← Normal mode angular frequency.

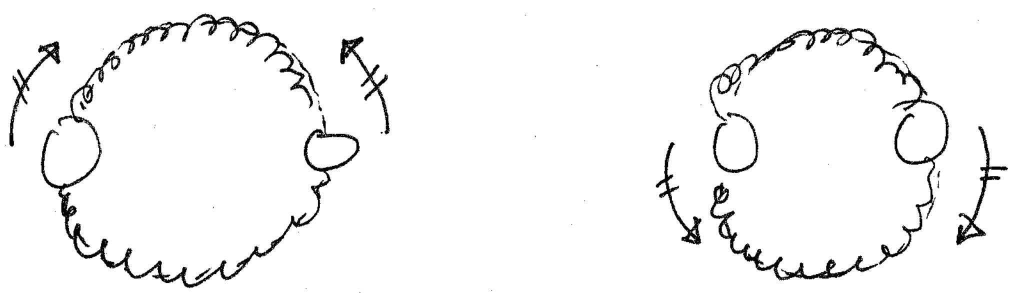
$$\boxed{V_2 \equiv \tilde{C}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega_0 t - \phi)}$$

• Normal mode #1 describes pure rotation of the 2 particles in a circle, with angular ~~speed~~ velocity $\tilde{V}_0$:



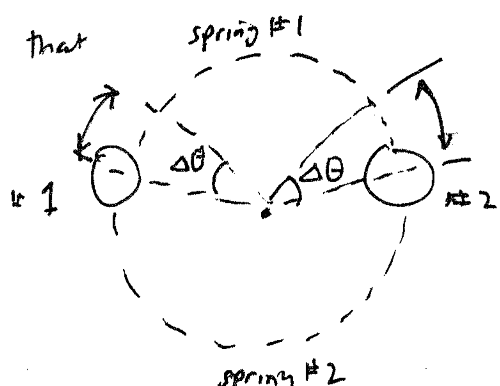
The springs between the 2 particles remain unstretched / not compressed relative to equilibrium configuration at all times.

• Normal mode #2 describes anti-symmetric oscillation of 2 particles.



2 particles Move towards each other w/ same angular speed.

Notice that

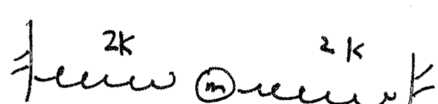


when #1 moves by $\Delta\theta$, so does #2.

$\Rightarrow$ spring #1 compressed by $2(\Delta\theta)R$.

Also, spring #2 is stretched by $2(\Delta\theta)R$. at the same time.

$\Rightarrow$ so this is equivalent to the following one particle system:

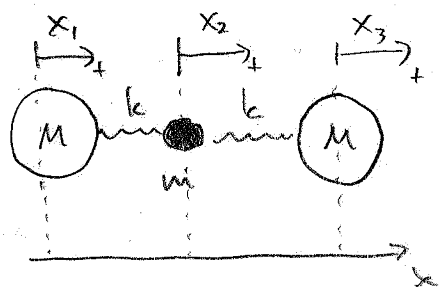


$$\Rightarrow \boxed{\omega_2 = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} = 2\omega_0}.$$

Problem 2 Solution

$x_n \equiv$ displacement from equilibrium

a.



$@\, x_1 = 0 \Rightarrow$ corresponding mass is at equilibrium (no net forces).

Whenever @ equilibrium, we define the displacement as <u>zero</u>.

b. Atom 1: $\sum F_1 = M\ddot{x}_1 = k(x_2 - x_1)$

if $x_2 - x_1 > 0 \Rightarrow$ spring is stretched $\Rightarrow$ restoring force in + direction

Atom 2: $\sum F_2 = m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$

Atom 3: $\sum F_3 = M\ddot{x}_3 = -k(x_3 - x_2)$

if $x_3 - x_2 > 0 \Rightarrow$ spring is stretched $\Rightarrow$ restoring force in (−) direction

$$\text{EOM}_1: \quad \ddot{x}_1 + \frac{k}{M}(x_1 - x_2) = 0$$

$$\text{EOM}_2: \quad \ddot{x}_2 + \frac{k}{m}(-x_1 + 2x_2 - x_3) = 0$$

$$\text{EOM}_3: \quad \ddot{x}_3 + \frac{k}{M}(-x_2 + x_3) = 0$$

$$\omega_1 \equiv \sqrt{\frac{k}{M}} \qquad \omega_2 \equiv \sqrt{\frac{k}{m}}$$

c.

$$-\omega_1^2 x_1 + \omega_1^2 x_2 + 0 \cdot x_3 = \ddot{x}_1$$

$$\omega_2^2 x_1 - 2\omega_2^2 x_2 + \omega_2^2 x_3 = \ddot{x}_2$$

$$0 \cdot x_1 + \omega_1^2 x_2 - \omega_1^2 x_3 = \ddot{x}_3$$

from equations to matrix form

$$\underbrace{\begin{pmatrix} -\omega_1^2 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2 & \omega_2^2 \\ 0 & \omega_1^2 & -\omega_1^2 \end{pmatrix}}_{\text{matrix } P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix}$$

d. Guess: $z(t) = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\alpha t} \Rightarrow \ddot{z} = -\alpha^2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\alpha t}$

Plug into complex EOM

$$\Rightarrow \underbrace{P \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cancel{e^{i\alpha t}}}_{z(t)} = \underbrace{-\alpha^2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cancel{e^{i\alpha t}}}_{\ddot{z}(t)}$$

used identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow P \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = -\begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Rightarrow P \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} + \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

$\vec{0} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left[\begin{pmatrix} -w_1^2 & w_1^2 & 0 \\ w_2^2 & -2w_2^2 & w_2^2 \\ 0 & w_1^2 & -w_1^2 \end{pmatrix} + \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \right] \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

$$\Rightarrow \underbrace{\begin{pmatrix} -w_1^2+\alpha^2 & w_1^2 & 0 \\ w_2^2 & -2w_2^2+\alpha^2 & w_2^2 \\ 0 & w_1^2 & -w_1^2+\alpha^2 \end{pmatrix}}_{\equiv P_1} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

where $\det(P_1) = 0$, we can find non-trivial sol'ns b/c it means P_1 has no inverse P_1^{-1}

Remember, trivial sol'n is when $A=B=C=0$, and this is not very useful

$$\det(P_1) = (-w_1^2+\alpha^2) \begin{vmatrix} -2w_2^2+\alpha^2 & w_2^2 \\ w_1^2 & -w_1^2+\alpha^2 \end{vmatrix} - w_1^2 \begin{vmatrix} w_2^2 & w_2^2 \\ 0 & -w_1^2+\alpha^2 \end{vmatrix} = 0$$

If $\det(P_1) = 0 \Rightarrow$ matrix cannot be inverted (P_1^{-1} does not exist).

If P_1 existed, then $P_1^{-1} P \begin{pmatrix} A \\ B \\ C \end{pmatrix} = P_1^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ← not useful

$$0 = \det(P_1) = (-w_1^2 + \alpha^2)\left[(-2w_2^2 + \alpha^2)(-w_1^2 + \alpha^2) - w_1^2 w_2^2\right] - w_1^2\left[w_2^2(-w_1^2 + \alpha^2) - 0\right]$$

$$= (-w_1^2 + \alpha^2)\left[w_1^2 w_2^2 - (2w_2^2 + w_1^2)\alpha^2 + \alpha^4\right] - w_1^2\left[-w_1^2 w_2^2 + w_2^2\alpha^2\right] = 0$$

$$0 = -\cancel{w_1^4 w_2^2} + (2w_1^2 w_2^2 + w_1^4)\alpha^2 - w_1^2\alpha^4 + \cancel{w_1^2 w_2^2 \alpha^2} - (2w_2^2 + w_1^2)\alpha^4 + \alpha^6 + \cancel{w_1^4 w_2^2} - \cancel{w_1^2 w_2^2 \alpha^2}$$

$$= \alpha^6 - (2w_2^2 + 2w_1^2)\alpha^4 + (2w_1^2 w_2^2 + w_1^4)\alpha^2 = 0 \quad \leftarrow \boxed{\text{Eqn } 1}$$

$$\Rightarrow \alpha^4 - (2w_2^2 + 2w_1^2)\alpha^2 + (2w_1^2 w_2^2 + w_1^4) = 0$$

; let some $x = \alpha^2$

$$\Rightarrow x^2 - 2(w_1^2 + w_2^2)x + (2w_1^2 w_2^2 + w_1^4) = 0$$

← we can use the quadratic equation to find x

$$\frac{2(w_1^2 + w_2^2) \pm \sqrt{4(w_1^4 + 2w_1^2 w_2^2 + w_2^4) - 4(2w_1^2 w_2^2 + w_1^4)}}{2}$$

$$= \frac{\cancel{2}w_1^2 + w_2^2 \pm \cancel{2}\sqrt{\cancel{w_1^4} - \cancel{w_1^4} + \cancel{2w_1^2 w_2^2} - \cancel{2w_1^2 w_2^2} + w_2^4}}{\cancel{2}}$$

$$\alpha^2 = w_1^2 + w_2^2 \pm w_2^2 = x$$

$$\Rightarrow \alpha^2 = w_1^2 \quad \text{or} \quad \alpha^2 = w_1^2 + 2w_2^2$$

$$\Rightarrow \boxed{\alpha = \pm w_1} \quad \text{or} \quad \boxed{\alpha = \pm\sqrt{w_1^2 + 2w_2^2}}$$

} 4 solutions, but Eqn 1 suggests there should be 6,

well $\alpha^2 = 0$ is also a solution

$$\Rightarrow \underbrace{\boxed{\alpha = 0} \text{ or } \boxed{\alpha = 0}}_{\text{2 more solutions}}$$

⇒ There are 5 possible values for α (six if you take redundancy into account), which makes sense given α goes to the order of 6 (α^6) in Eqn 1

e. $\alpha = 0$ does not correspond to a vibrational mode.

In our guess for $z(t)$, $e^{i0t} = 1$ $\Rightarrow$ no oscillation

$\Rightarrow \alpha = 0$ corresponds to a translational normal mode where all three atoms displace in the same direction with equal constant velocity.

$$V_5 \equiv \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i0t} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad \text{in (f.) we find that } A = B = C$$

$$\Rightarrow V_5 \equiv A_5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad [A_5] = \text{length}$$

We could also guess $z(t) = t \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ to be a solution to the EOM, and we'd find it to be true, and you'd also find that $A = B = C$ (you can try it to prove it to yourself..)

$$\Rightarrow V_6 \equiv B_6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t \qquad [B_6] = \frac{\text{length}}{\text{time}}$$

$\Rightarrow$ the normal mode associated with $\alpha = 0$ can be written as follows:

we will call this α_3, $\alpha_3 = 0$

$$V_5 + V_6 = N_{\alpha_3}$$

f. $\alpha = \pm \omega_1$ ← we examine $\alpha = \pm\omega_1$ b/c $\alpha^2 = \omega_1^2$ for either value of α

$$P_1 = \begin{pmatrix} 0 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2+\omega_1^2 & \omega_2^2 \\ 0 & \omega_1^2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2+\omega_1^2 & \omega_2^2 \\ 0 & \omega_1^2 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

$$\Rightarrow B\omega_1^2 = 0 \; ; \; A\omega_2^2 + (-2\omega_2^2+\omega_1^2)B + C\omega_2^2 = 0 \; ; \; B\omega_1^2 = 0$$

$$\Downarrow$$

$$B = 0 \Rightarrow A\omega_2^2 + C\omega_2^2 = 0 \Rightarrow A = -C, \; C = -A$$

so, $\boxed{B = 0, \; C = -A}$

$\alpha = \pm\sqrt{\omega_1^2 + 2\omega_2^2}$

$$P_1 = \begin{pmatrix} 2\omega_2^2 & \omega_1^2 & 0 \\ \omega_2^2 & \omega_1^2 & \omega_2^2 \\ 0 & \omega_1^2 & 2\omega_2^2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2\omega_2^2 & \omega_1^2 & 0 \\ \omega_2^2 & \omega_1^2 & \omega_2^2 \\ 0 & \omega_1^2 & 2\omega_2^2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

$$\Rightarrow 2\omega_2^2 A + \omega_1^2 B = 0 \; ; \; A\omega_2^2 + B\omega_1^2 + C\omega_2^2 = 0 \; ; \; B\omega_1^2 + 2\omega_2^2 C = 0$$

$$B = -\frac{2\omega_2^2 A}{\omega_1^2} \Rightarrow -A\omega_2^2 + C\omega_2^2 = 0$$

$$\Rightarrow C = A$$

substitute for ω_1^2, ω_2^2

$$B = -\frac{2M}{m}A \; ; \; M > m \Rightarrow |B| > |A|$$

$$\Rightarrow \boxed{B = -\frac{2\omega_2^2}{\omega_1^2}A = -\frac{2M}{m}A, \; C = A}$$

f. $\underline{\alpha = 0}$

$$P_1 = \begin{pmatrix} -\omega_1^2 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2 & \omega_2^2 \\ 0 & \omega_1^2 & -\omega_1^2 \end{pmatrix} \Rightarrow \begin{pmatrix} -\omega_1^2 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2 & \omega_2^2 \\ 0 & \omega_1^2 & -\omega_1^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \vec{0}$$

$$\Rightarrow -A\omega_1^2 + B\omega_1^2 = 0 \quad \Rightarrow A = B$$

$$\Rightarrow B\omega_1^2 - C\omega_1^2 = 0 \quad \Rightarrow B = C$$

$$\Rightarrow \boxed{A = B = C}$$

$$\Rightarrow A\omega_2^2 - 2B\omega_2^2 + C\omega_2^2 = A\omega_2^2 - 2A\omega_2^2 + A\omega_2^2 = 0$$

g. from part (f), we see we can write the following for cases were $\alpha = \omega_1$ + $\alpha = -\omega_1$

$$V_1 \equiv A_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{i\omega_1 t}$$

b/c $A = -C$, $B = 0$
we can set A_1, thus it is a free parameter

$$V_2 \equiv A_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-i\omega_1 t}$$

A_2 is also a free parameter

$\Rightarrow$ normal mode associated with $\alpha = \pm\omega_1$ ← call this α_1, $\alpha_1 = \pm\omega_1$

$$V_1 + V_2 = N_{\alpha_1}$$

for $\alpha = \pm\sqrt{\omega_1^2 + 2\omega_2^2}$

$$V_3 = A_3 \begin{pmatrix} 1 \\ \frac{-2\omega_2^2}{\omega_1^2} \\ 1 \end{pmatrix} e^{i\left(\sqrt{\omega_1^2+2\omega_2^2}\right)t}$$

b/c $A = C$, $B = \frac{-2\omega_2^2}{\omega_1^2} A$

$$V_4 = A_4 \begin{pmatrix} 1 \\ \frac{-2\omega_2^2}{\omega_1^2} \\ 1 \end{pmatrix} e^{-i\left(\sqrt{\omega_1^2+2\omega_2^2}\right)t}$$

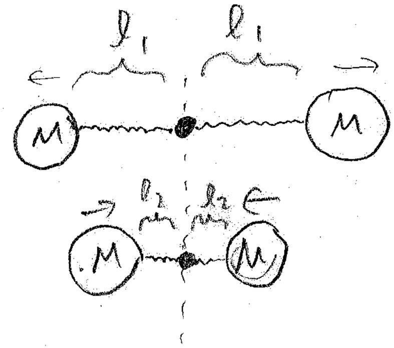
A_3 + A_4 are free parameters

Now, let's draw what these look like ...

for $\alpha = \pm \omega_1$

snapshot 1 :

snapshot 2 :



all oscillating with angular frequency of magnitude ω_1

Describes motion where the middle mass is stationary & two outer masses oscillate in opposite directions with equal amplitude

$$e^{i\omega_1 t} = \cos(\omega_1 t) + i\sin(\omega_1 t)$$
$$e^{-i\omega_1 t} = \cos(\omega_1 t) - i\sin(\omega_1 t)$$

the real parts of both solutions are equal

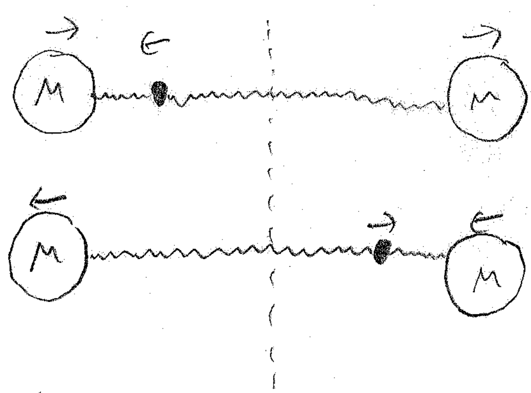
$\Rightarrow$ If we were to look at both cases $\alpha = \omega_1$ & $\alpha = -\omega_1$ independently, we would not be able to see a difference in their motion

i.e. - one at a time

for $\alpha = \pm\sqrt{\omega_1^2 + 2\omega_2^2}$

snapshot 1 :

snapshot 2 :

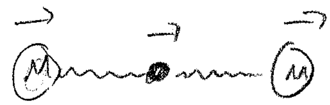


all oscillating with angular frequency $\sqrt{\omega_1^2 + 2\omega_2^2}$

Describes a vibrational mode where the two outer masses oscillate in the same direction with equal amplitude. The middle mass oscillates in the opposite direction with increased amplitude by a factor of $\frac{2M}{m}$. All oscillate with angular frequency given by $\alpha = \sqrt{\omega_1^2 + 2\omega_2^2}$

There will be no observable difference between cases where $\alpha = \sqrt{\omega_1^2 + 2\omega_2^2}$ or $\alpha = -\sqrt{\omega_1^2 + 2\omega_2^2}$

for $\alpha = 0$

snapshot 1: 

← no oscillation

Describes a translational mode where the three masses displace with equal constant velocity (in same direction)

$\Rightarrow$ normal mode associated with $\alpha = \underbrace{\pm\sqrt{\omega_1^2+2\omega_2^2}}$

we will call this α_2, $\alpha_2 = \pm\sqrt{\omega_1^2+2\omega_2^2}$

$$V_3 + V_4 = N_{\alpha_2}$$

Thus, we have 3 normal modes: N_{α_1}, N_{α_2}, + N_{α_3}

We expect to have 3 normal modes because we have 3 equations of motion.

Each equation of motion is a second-order linear differential equation, i.e. the highest derivative is the second derivative, thus we expect to have 2 free parameters for each of the 3 solutions to the 3 equations of motion $\Rightarrow$ 6 total free parameters

$N_{\alpha_1} = V_1 + V_2$, V_1 + V_2 each have a free parameter: A_1, A_2, respectively

$N_{\alpha_2} = V_3 + V_4$, V_3 + V_4 each have a free parameter: A_3, A_4, respectively

$N_{\alpha_3} = V_5 + V_6$, V_5 + V_6 each have a free parameter: A_5, B_6, respectively

$\Rightarrow$ general solution is the superposition of normal modes:

$$\boxed{z_g(t) = N_{\alpha_1} + N_{\alpha_2} + N_{\alpha_3} = V_1 + V_2 + V_3 + V_4 + V_5 + V_6}$$

Problem 3

FROM MIDTERM SOLUTIONS

(3-1)

1(a) The $\mathbb{C}$-equivalent EOM is : $\ddot{Z} + 2\gamma \dot{Z} + \omega_0^2 Z = \frac{F_0}{m}\cos(\omega t)$,

where $Z(t)$ is a $\mathbb{C}$-valued function.

1(b) Before solving, represent : $\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$:

$$\Rightarrow \quad \ddot{Z} + 2\gamma \dot{Z} + \omega_0^2 Z = \frac{F_0}{2m}\left\{ e^{i\omega t} + e^{-i\omega t} \right\}.$$

Then, let's solve : $\ddot{Z}_1 + 2\gamma \dot{Z}_1 + \omega_0^2 Z_1 = \frac{F_0}{2m} e^{i\omega t}$

by guessing $Z_1(t) = A e^{i\omega t}$ ← Guess

Plugging in Check : $\ddot{Z}_1 + 2\gamma \dot{Z}_1 + \omega_0^2 Z_1$

$$= -\omega^2 A e^{i\omega t} + 2\gamma i\omega e^{i\omega t} A + \omega_0^2 A e^{i\omega t}$$

$$\Rightarrow \quad A\cancel{e^{i\omega t}}\left[-\omega^2 + 2\gamma i\omega + \omega_0^2\right] = \frac{F_0}{2m}\cancel{e^{i\omega t}}$$

want

$\Rightarrow$

Need : $$\boxed{A = \frac{F_0}{2m\left[-\omega^2 + 2i\gamma\omega + \omega_0^2\right]}}$$

~~(And for~~

1(c) (3-2)

Linearity: If z_1 is sol'n to EOM, and so is z_2, then $z_1 + z_2$ is also a solution to EOM.

This is useful to us since:

$z_1(t)$ (found in (b)) is solution to:

$$\ddot{z}_1 + 2\gamma \dot{z}_1 + \omega_0^2 z_1 = \frac{F_0}{2m} e^{i\omega t}$$

and if we have $z_2(t)$ being solution to: (we can get $z_2(t)$ from z_1 by just changing $\omega \to -\omega$

$$\ddot{z}_2 + 2\gamma \dot{z}_2 + \omega_0^2 z_2 = \frac{F_0}{2m} e^{-i\omega t}$$

then $z_p \equiv z_1 + z_2$ satisfies:

$$\ddot{z}_p + 2\gamma \dot{z}_p + z_p \omega_0^2 = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \}$$

$\uparrow$ our original $\mathbb{C}$-equivalent eq'n of motion

But $z_p(t)$ has no free parameters.

~~Next~~ But note that:

$$\{ \ddot{z}_{d\,hm} + 2\gamma \dot{z}_{d\,hm} + \omega_0^2 z_{d\,hm} = 0 \}$$

and

$$+ \{ \ddot{z}_p + 2\gamma \dot{z}_p + z_p \omega_0^2 = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \} \}$$

$$\rightarrow \quad \ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = \frac{F_0}{2m} \{ e^{i\omega t} + e^{-i\omega t} \}$$

where $\boxed{z = z_{d\,hm} + z_p}$ $\leftarrow$ $z_{d\,hm}(t) = e^{-\gamma t} \{ A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t} \}$

$\uparrow$ found in Q#2.

So, using linearity again, we get the general sol'n: (3-3)

$$\boxed{\begin{aligned} Z(t) &= e^{-\gamma t}\left\{ A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t} \right\} \\ &+ \frac{F_0 e^{i\omega t}}{2m}\left\{ \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2} \right\} \\ &+ \frac{F_0 e^{-i\omega t}}{2m}\left\{ \frac{1}{-\omega^2 - 2i\gamma\omega + \omega_0^2} \right\} \end{aligned}}$$

where $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2}$. A & B are free parameter.

(part f. on Pset #2)

(d) $$X_p(t) = \frac{F_0}{m} \frac{\left\{ \Omega^2 \cos(\omega t) + 2\gamma\omega \sin(\omega t) \right\}}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (1)$$

want to write as $X_p(t) = D(\omega)\cos(\tilde{\omega} t - \theta)$

$$= D(\omega)\cos(\tilde{\omega} t)\cos(\theta) + D(\omega)\sin(\tilde{\omega} t)\sin(\theta)$$

So, matching $\sin(\omega t)$ and $\cos(\omega t)$ terms in eq'n (1) with these:

We have:

$$D(\omega)\cos\theta = \frac{F_0}{m}\frac{\Omega^2}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (2)$$

$$D\sin\theta = \frac{F_0}{m}\frac{2\gamma\omega}{\Omega^4 + 4\gamma^2\omega^2} \quad \ldots (3)$$

$(2)^2 + (3)^2$:

$$D^2 = \left(\frac{F_0}{m}\right)^2 \frac{\Omega^4 + 4\gamma^2\omega^2}{(\Omega^4 + 4\gamma^2\omega^2)^2} = \left(\frac{F_0}{m}\right)^2 \frac{1}{\Omega^4 + 4\gamma^2\omega^2}$$

$$\Rightarrow \boxed{D(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\Omega^4 + 4\gamma^2\omega^2}}}$$

over →

To maximize $D(\omega)$: minimize the denominator

3-4

$$\Omega^4 + 4\gamma^2\omega^2 .$$

$$\Omega \equiv \sqrt{\omega_0^2 - \omega^2}$$

$$\Rightarrow \quad 0 = \frac{d(\Omega^4 + 4\gamma^2\omega^2)}{d\omega}$$

$$= \frac{d}{d\omega}\left\{ (\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 \right\}$$

$$= 2(\omega_0^2 - \omega^2)(-2\omega) + 8\gamma^2\omega$$

$$\Rightarrow \quad -4\cancel{\omega}(\omega_0^2 - \omega^2) = -8\gamma^2\cancel{\omega}$$

($\omega \neq 0$ since we have oscillatory force)

$$\omega_0^2 - \omega^2 = 2\gamma^2$$

$$\Rightarrow \quad \omega^2 = \omega_0^2 - 2\gamma^2$$

$$\Rightarrow \quad \boxed{\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}}$$

← resonant angular frequency

(e) $t \gg \frac{1}{\gamma}$: Eq'n (5) on midterm handout becomes:

$$X(t) \approx X_p(t).$$

so: $$\boxed{E_{total}(t) = \frac{1}{2} k x_p(t)^2 + \frac{m\dot{x}_p^2}{2}}$$

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f. (actually part d. on Pset # 2)

We found $z(t)$ to be as follows:

$$z(t) = e^{-\gamma t}\left[Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right]$$ ← we know the real part is $Ce^{-\gamma t}\cos(\tilde{\omega}t - \phi)$ from our studies of under-damped harmonic oscillators $\Rightarrow x_h(t) = Ce^{-\gamma t}(\cos(\tilde{\omega}t-\phi))$

$$z_p(t) \equiv \underbrace{+ \frac{F_0 e^{i\omega t}}{2m}\left[\frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right] + \frac{F_0 e^{-i\omega t}}{2m}\left[\frac{1}{-\omega^2 - 2i\gamma\omega + \omega_0^2}\right]}$$

Finding $Re[z_p(t)]$ will be a little tricky because we have imaginary components (i) in the denominator. To remove i from the denominator, we will use a common strategy of multiplying the top + bottom of each term by what's called the complex conjugate...

A little background: $n^* \equiv$ complex conjugate of n

$i^* = -i$ (by definition)

if a is real, $a^* = a$

$$\Rightarrow \left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right)^* = -\omega^2 - 2i\gamma\omega + \omega_0^2 = \Omega^2 - 2i\gamma\omega$$

$\boxed{\Omega \equiv \omega_0^2 - \omega^2}$

$$\Rightarrow \left(-\omega^2 + (-2i\gamma\omega) + \omega_0^2\right)^* = -\omega^2 + 2i\gamma\omega + \omega_0^2 = \Omega^2 + 2i\gamma\omega$$

$$\Rightarrow z_p(t) = \frac{F_0 e^{i\omega t}}{2m}\left[\frac{\Omega^2 - 2i\gamma\omega}{(\Omega^2 + 2i\gamma\omega)(\Omega^2 - 2i\gamma\omega)}\right]$$ ← multiplied by $\boxed{\frac{\Omega^2 - 2i\gamma\omega}{\Omega^2 - 2i\gamma\omega} = 1}$

$$+ \frac{F_0 e^{-i\omega t}}{2m}\left[\frac{\Omega^2 + 2i\gamma\omega}{(\Omega^2 - 2i\gamma\omega)(\Omega^2 + 2i\gamma\omega)}\right]$$ ← multiplied by $\boxed{\frac{\Omega^2 + 2i\gamma\omega}{\Omega^2 + 2i\gamma\omega} = 1}$

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$$\Rightarrow z_p(t) = \frac{F_0 e^{i\omega t}}{2m}\left[\frac{\Omega^2 - 2i\gamma\omega}{\Omega^4 + 4\gamma^2\omega^2}\right] + \frac{F_0 e^{-i\omega t}}{2m}\left[\frac{\Omega^2 + 2i\gamma\omega}{\Omega^4 + 4\gamma^2\omega^2}\right]$$

common denominator (b/c two original denominators were complex conjugates of each other)

$$\Rightarrow z_p(t) = \frac{F_0}{2m(\Omega^4+4\gamma^2\omega^2)}\left[\Omega^2(e^{i\omega t}+e^{-i\omega t}) + 2i\gamma\omega(e^{-i\omega t}-e^{i\omega t})\right]$$

$$\Rightarrow z_p(t) = \frac{F_0}{2m(\Omega^4+4\gamma^2\omega^2)}\left[\Omega^2(2\cos\omega t) + 2i\gamma\omega(-2i\sin\omega t)\right]$$

$$\Rightarrow x_p(t) = \frac{F_0}{\cancel{2}m(\Omega^4+4\gamma^2\omega^2)}\left[\Omega^2(\cancel{2}\cos\omega t) + \overset{2}{\cancel{4}}\gamma\omega\sin\omega t\right]$$

$$\Rightarrow x_p(t) = \frac{F_0[\Omega^2\cos\omega t + 2\gamma\omega\sin\omega t]}{m[\Omega^4+4\gamma^2\omega^2]}$$

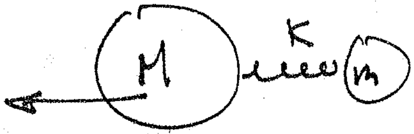
this is real, so we can call it $x_p(t)$

$\Rightarrow$ to find real general solution, add $x_h(t) + x_p(t)$

$$\Rightarrow \boxed{x(t) = Ce^{-\gamma t}\cos(\tilde{\omega}t - \phi) + \frac{F_0}{m}\frac{[\Omega^2\cos\omega t + 2\gamma\omega\sin\omega t]}{[\Omega^4+4\gamma^2\omega^2]}}$$

Pset 2 : Problem 4

$M >> m$ (M is much larger mass than m)



$F(t) = F_0 \cos(\omega t)$.

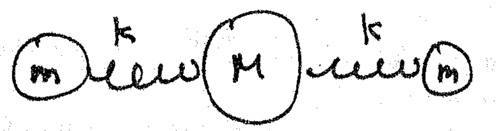
Let $\omega_0 \equiv \sqrt{\frac{K}{m}}$

Suppose $\underline{\omega = \omega_0}$. We want to show that if $\omega = \omega_0$, then the big particle (of mass M) will be stationary.

The key in this problem is to realize that we can have whatever physical mechanism providing the oscillatory force F(t). (as long as F(t) oscillates with angular frequency $\omega = \omega_0$).

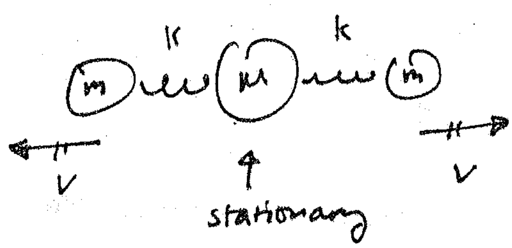
One way to do this is by <u>attaching another</u> m to the left side of M, using ~~spr~~ <u>another spring</u> of spring constant K.

So :



To provide the oscillatory force, pull the left m by an amplitude A; then release it. ~~then the left m oscillates, it oscillates~~

In the normal mode where we have the 2 m's oscillating antisymmetrically with respect to each other, M remains stationary.



The force that the left m exert on M is:

$$F(t) = F_0 \cos(\omega_0 t)$$
$$= KA \cos(\omega_0 t)$$

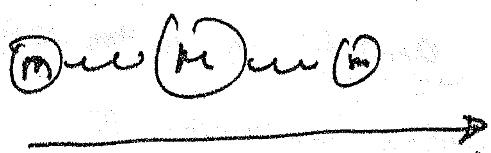
(K ≡ spring constant)

since left m oscillates with frequency ω_0 in this normal mode, with amplitude A. Since A can be any amplitude we desire, we can exert any ~~force~~ amplitude of force F_0 ...

From problem 2, you found that there are 3 normal modes.

One of them describes the normal mode with M stationary, described on previous page. We now need to show that M is stationary in the other 2 normal modes, then we'd be done w/ this problem. (since general motion of M is sum of the 3 normal modes, and thus if M doesn't move in all 3 normal modes, then the sum of them also describes M being stationary.)

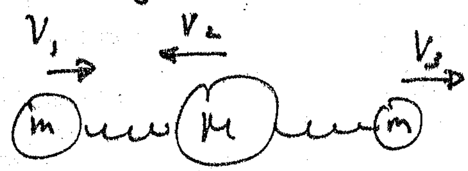
In another normal mode you found in problem 2:



translational motion (No oscillations).

We discount this possibility by assuming that such translational motion is forbidden in this engineering systems. (i.e. we're interested in knowing if M oscillates relative to its right side m.)

Finally, the only other normal mode describes antisymmetric oscillation of following type:



In your solution Problem 2, you should have found that if $M \gg m$, then $V_2 \ll V_1$ and $V_2 \ll V_3$. $\Rightarrow$ M is approximately stationary

$\therefore$ $\boxed{\text{We've shown that } M \text{ remains stationary.}}$