MITES 08 : Physics III : Solution set 4

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(pg 1)

1.) Expanding Wall.

(a)

$$\frac{\partial^2 \tilde{y}}{\partial t^2} = v^2 \frac{\partial^2 \tilde{y}}{\partial x^2}$$ ← $\mathbb{C}$-equivalent wave equation

Valid for $0 \le x \le L$. (Between 2 walls)

Guess: $\tilde{y}(x,t) = [Ae^{i(kx-\omega t)} + Be^{i(kx-\omega t)}]$

right moving plane wave → ; ← left moving plane wave.

$= [Ae^{ikx} + Be^{-ikx}]\, e^{-i\omega t}$

A, B constants.

$k = 2\pi/\lambda$.

k yet to be specified a value.

· We already know $\tilde{y}(x,t)$ is a solution to wave eqn. since it has the form

$y(x,t) = f(x-vt) + g(x+vt)$

$(\omega = kv)$.

· Boundary condition 1 :

$\tilde{y}(0,t) = 0 = [Ae^{-i\omega t} + Be^{-i\omega t}]$

$= e^{-i\omega t}(A+B)$

$\Rightarrow \boxed{A = -B}$

· so ∴ $\tilde{y}(x,t) = A(e^{ikx} - e^{-ikx})\, e^{-i\omega t}$

$= 2iA \sin(kx)\, e^{-i\omega t}$

Next, Boundary Condition 2 : $\tilde{y}(L,t) = 0 = 2Ai\sin(kL)\, e^{-i\omega t}$

$\Rightarrow \sin(kL) = 0$ ($A \neq 0$ since if $A=0$, then $\tilde{y}(x,t) = 0$ ~~for~~ all x.

$\Rightarrow kL = n\pi$ $n = 1, 2, 3, \ldots$ ← trivial, uninteresting solution.)

$\Rightarrow \boxed{k_n = \frac{n\pi}{L}}$

For the same reason, ignore $n=0$.

$$\therefore \quad \tilde{y}_n(x,t) = 2Ai\sin\left(\frac{n\pi x}{L}\right)e^{-i\omega_n t}$$

$$\omega_n = k_n V = \frac{n\pi v}{L}$$

$$= 2Ai\sin\left(\frac{n\pi x}{L}\right)\left\{\cos(\omega_n t) - i\sin(\omega_n t)\right\}$$

$$= \left\{\underbrace{2A}_{=\tilde{A}}\sin(\omega_n t) + \underbrace{2Ai}_{=\tilde{B}}\cos(\omega_n t)\right\}\sin\left(\frac{n\pi x}{L}\right)$$

← constants relabeled.

$$= \left\{\tilde{A}\sin(\omega_n t) + \tilde{B}\cos(\omega_n t)\right\}\sin\left(\frac{n\pi x}{L}\right)$$

From above, we see that the real valued solution is:

$$\boxed{y_n(x,t) = \left\{A_n\sin\left(\frac{n\pi v t}{L}\right) + B_n\cos\left(\frac{n\pi v t}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right)}$$

$n = 1, 2, 3, \ldots$

(b) At $t=0$: ① $\phi(x) = -\alpha\left\{\frac{L^2}{4} + \left(x - \frac{L}{2}\right)^2\right\}$ $(0 \le x \le L)$ $= y(x, t=0)$

2 initial conditions → ② And $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0$ ∵ string initially at rest.

Fourier sine series:

$$\phi(x) = y(x, t=0) = \sum_{n=1}^{\infty} B_n \overbrace{\cos\left(\frac{n\pi v t}{L}\right)}^{1} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Want to figure out what A_n and B_n are from the 2 initial conditions..

First:
$$0 = \frac{\partial y(x,t)}{\partial t}\bigg|_{t=0} = \sum_{n=1}^{\infty} \frac{\partial y_n}{\partial t}\bigg|_{t=0}$$
$$= \sum_{n=1}^{\infty} \left\{ \omega_n \underbrace{A_n \cos(\omega_n t)}_{=1 \text{ when } t=0} - \omega_n \underbrace{B_n \sin(\omega_n t)}_{=0 \text{ when } t=0} \right\} \sin\left(\frac{n\pi x}{L}\right)\bigg|_{t=0}$$
$$= \sum_{n=1}^{\infty} \underbrace{\omega_n A_n}_{\equiv C_n} \sin\left(\frac{n\pi x}{L}\right)$$
$\equiv C_n$ ← constant. relabeled.

$$\Rightarrow \quad 0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad C_n = \frac{2}{L}\int_0^L 0 \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$= 0.$ for all n. $\Rightarrow$ $\omega_n A_n = 0$ for all n.

$\Rightarrow$ $\boxed{A_n = 0}$ for all n.

Next, find B_n's:

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$
$$\Rightarrow \quad B_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{-2\alpha}{L}\int_0^L \left\{ \frac{L^2}{4} + \left(x-\frac{L}{2}\right)^2 \right\} \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{-2\alpha}{L}\frac{L^2}{4} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx}_{= -\frac{L}{n\pi}\cos\left(\frac{n\pi L}{L}\right) + \frac{L}{n\pi} \;=\; \frac{L}{n\pi}\left(1-(-1)^n\right)} - \frac{2\alpha}{L}\int_0^L dx \left(x-\frac{L}{2}\right)^2 \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad B_n = -\frac{\alpha L}{2}\cdot\frac{L}{n\pi}\left[1-(-1)^n\right] - \frac{2\alpha}{L}\int_0^L dx\,\left(x-\frac{L}{2}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \quad \leftarrow \boxed{\text{Eqn (1)}}$$

Now,

$$\int_0^L dx\,\left(x-\frac{L}{2}\right)^2 \sin\left(\frac{n\pi x}{L}\right) = \frac{-L}{n\pi}\left(x-\frac{L}{2}\right)^2\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{L\,2}{n\pi}\int_0^L dx\,\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{-L}{n\pi}\frac{L^2}{4}(-1)^n + \frac{L}{n\pi}\frac{L^2}{4}$$

$$+ \frac{2L}{n\pi}\left\{ \cancel{\frac{L}{n\pi}\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{0}\left(x-\frac{L}{2}\right)\Bigg|_0^L} - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= \frac{L^3}{4n\pi}\left[(-1)^{n+1}+1\right] + \frac{2L}{n\pi}\left\{ \cancel{\frac{L^2}{2n\pi}}\,0 - 0 - \frac{L}{n\pi}\left[\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L \right\}$$

$$= \frac{L^3}{4n\pi}\left[(-1)^{n+1}+1\right] + \frac{2L}{n\pi}\left\{ \left(\frac{L}{n\pi}\right)^2\left[\underbrace{\cos(n\pi)}_{(-1)^n} - 1\right]\right\}$$

$$= \frac{L^3}{4n\pi}\left[(-1)^{n+1}+1\right] + \frac{2L^3}{(n\pi)^3}\left[(-1)^n-1\right]$$

∴ Eqn (1) at the top becomes:

$$B_n = +\frac{\alpha L^2}{2n\pi}\left[(-1)^n-1\right] - \frac{2\alpha}{L}\left(\frac{L^3}{4n\pi}\right)\left[(-1)^{n+1}+1\right] - \frac{2\alpha}{L}\left(\frac{2L^3}{(n\pi)^3}\right)\left[(-1)^n-1\right]$$

$$= \frac{\alpha L^2}{2n\pi}\left[(-1)^n-1\right] + \frac{\alpha L^2}{2n\pi}\left[(-1)^n-1\right] - \frac{4\alpha L^2}{(n\pi)^3}\left[(-1)^n-1\right]$$

$$= \left[(-1)^n-1\right]\left\{\frac{\alpha L^2}{n\pi} - \frac{4\alpha L^2}{(n\pi)^3}\right\}$$

$$= \boxed{\begin{cases} 0 & \text{if } n = 2,4,6,8,\dots \text{ (even)} \\ -\dfrac{2\alpha L^2}{n\pi}\left[1-\dfrac{4}{(n\pi)^2}\right] & \text{if } n = 1,3,5,\dots \text{ (odd.)} \end{cases}}$$

1(c) From (b), we found that $A_n = 0$ for all n.

and, $B_n = \begin{cases} 0 & \text{if } n = 2,4,6,8,\ldots \text{ (even)} \\ -\dfrac{2\alpha L^2}{n\pi}\left[1 - \dfrac{4}{(n\pi)^2}\right] & \text{if } n = 1,3,5,\ldots \text{ (odd)} \end{cases}$

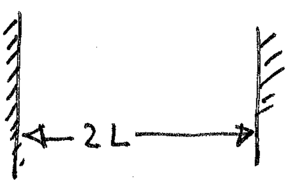
$$\therefore\ y(x, t=0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{-2\alpha L^2}{\pi} \sum_{\substack{n=1,3,5,7,\ldots \\ \text{(odd)}}} \frac{1}{n}\left[\frac{4}{(n\pi)^2} - 1\right] \sin\left(\frac{n\pi x}{L}\right)$$

$\therefore$ For $t > 0$: subsequent shape of string $y(x,t)$ after releasing the string at $t = 0$ is:

$$\boxed{y(x,t) = \sum_{\substack{n=1,3,5,\ldots \\ \text{(odd)}}} \frac{-2\alpha L^2}{\pi} \frac{1}{n}\left[\frac{4}{(n\pi)^2} - 1\right] \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v t}{L}\right)}$$

1(d)



The new normal modes can be obtained from the older (previous ones) for walls of separation L by: changing $L \longrightarrow 2L$

Before: (walls separated by "L")

$$y_n(x,t) = \left\{ A_n \sin\left(\frac{n\pi v t}{L}\right) + B_n \cos\left(\frac{n\pi v t}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right)$$

change $L \to 2L$ wherever you see "L" in above equation.

Normal modes for 2 walls separated by $2L$.

$$\boxed{y_n(x,t) = \left\{ A_n \sin\left(\frac{n\pi v t}{2L}\right) + B_n \cos\left(\frac{n\pi v t}{2L}\right) \right\} \sin\left(\frac{n\pi x}{2L}\right)}$$

$n = 1,2,3,\ldots$

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$ (pg6)

So now, initial conditions are:

① $\left.\dfrac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0$ (string initially at rest.)

② $\phi(x) = y(x, t=0) = \displaystyle\sum_{n=1}^{\infty} y_n(x,0)$

① again results in $A_n = 0$ as before.

② gives $\phi(x) = \displaystyle\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2L}\right)$

$$\Rightarrow B_n = \left(\frac{2}{2L}\right)\int_0^{2L} \phi(x) \sin\left(\frac{n\pi x}{2L}\right) dx$$

since $\phi(x) = 0$ for $L \le x \le 2L$

$$= \frac{1}{L}\int_0^L dx\, -\alpha\left\{\frac{L^2}{4} + \left(x-\frac{L}{2}\right)^2\right\} \sin\left(\frac{n\pi x}{2L}\right) dx$$

$$= \frac{-\alpha}{L}\left\{\frac{L^2}{4}\int_0^L dx \sin\left(\frac{n\pi x}{2L}\right) + \int_0^L dx \left(x-\frac{L}{2}\right)^2 \sin\left(\frac{n\pi x}{2L}\right)\right\}$$

$$= \frac{-\alpha}{L}\left\{\frac{L^2}{4}\left(\frac{-2L}{n\pi}\right)\cos\left(\frac{n\pi x}{2L}\right)\Big|_0^L + \left(\frac{-2L}{n\pi}\right)\cos\left(\frac{n\pi x}{2L}\right)\left(x-\frac{L}{2}\right)^2\Big|_0^L + \frac{4L}{n\pi}\int_0^L \left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{2L}\right) dx\right\}$$

$$= \frac{-\alpha}{L}\left\{\frac{L^2}{4}\left(\frac{-2L}{n\pi}\right)\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \frac{2L}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)\frac{L^2}{4} - \frac{L^2}{4}\right] + \frac{4L}{n\pi}\left[\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{2L}\right)\left(x-\frac{L}{2}\right)\Big|_0^L - \frac{2L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{2L}\right) dx\right]\right.$$

$$= \frac{-\alpha}{L}\left\{\frac{-L^3}{2n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \frac{L^3}{2n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] + \frac{8L^2}{(n\pi)^2}\left[\sin\left(\frac{n\pi}{2}\right)\frac{L}{2} + \frac{2L}{n\pi}\cos\left(\frac{n\pi x}{2L}\right)\Big|_0^L\right]\right.$$

$$= \frac{-\alpha}{L}\left\{\frac{-2L^3}{2n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] + \frac{8L^2}{(n\pi)^2}\left[\frac{L}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{2L}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]\right]\right\}$$

(over)

$$\Rightarrow \boxed{B_n = -\frac{\alpha}{L}\left\{ \left[\cos\left(\frac{n\pi}{2}\right)-1\right]\left[\frac{-L^3}{n\pi} + \frac{16L^3}{(n\pi)^3}\right] + \frac{4L^3}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right\}}$$

↓ Factor out some terms to make it look simpler:

$$B_n = -\frac{\alpha}{L}\,\frac{L^3}{n\pi}\left\{ \left[\cos\left(\frac{n\pi}{2}\right)-1\right]\left[-1 + \frac{16}{(n\pi)^2}\right] + \frac{4}{(n\pi)}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$= \boxed{-\frac{\alpha L^2}{n\pi}\left\{ \left[\cos\left(\frac{n\pi}{2}\right)-1\right]\left[-1 + \frac{16}{(n\pi)^2}\right] + \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right\}}$$

$$\therefore \boxed{y(x, t>0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2L}\right)\cos\left(\frac{n\pi v t}{2L}\right)}$$

(Where B_n was determined above.)

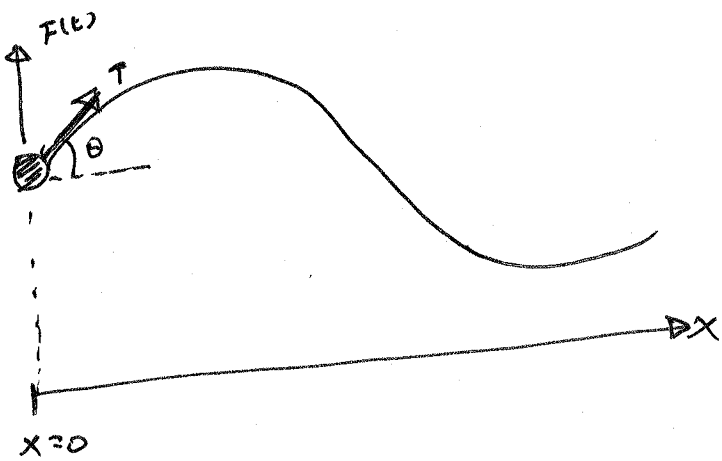
Problem 2: Characteristic Impedance of a string

By definition of impedance Z:

$$Z = \frac{F_0}{V_0}$$

Consider a particle at a specific location x on the vibrating string:

↳ described by $y(x,t) = A\sin(kx - \omega t)$.



Namely, consider a particle at position $x=0$.

Then

$$V(t) = \left.\frac{\partial y}{\partial t}\right|_{x=0}$$

$$= -A\omega\cos(kx-\omega t)\Big|_{x=0}$$

$$= -A\omega\cos(\omega t).$$

$$\Rightarrow \boxed{V_0 = A\omega}$$

← Maximum speed (in $\hat{y}$-direction) that the particle at $x=0$ can have.

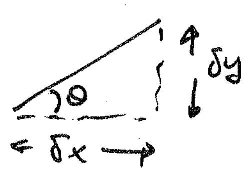
And $F(t) = T\sin\theta$

$\approx T\dfrac{\sin\theta}{\cos\theta}$ ← $\because |\theta| \ll 1$; (small vertical vibrations) so $\cos\theta \approx 1$.

$= T\tan\theta$ $\Rightarrow \sin\theta = \dfrac{\sin\theta}{1} \approx \dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

$= T\left.\dfrac{\partial y}{\partial x}\right|_{x=0}$

$= TAk\cos(kx-\omega t)\Big|_{x=0}$ since



$= (TAk)\cos(\omega t)$

⇓

$$F_0 = TAk$$

← maximum force on the vertically oscillating particle at $x=0$.

$$\therefore \quad Z = \frac{F_0}{V_0} = \frac{T\cancel{A}k}{\cancel{A}\omega} = \frac{Tk}{\omega}$$

but $c = \sqrt{\frac{T}{\rho}}$

and $kc = \omega$.

$$= \frac{T\cancel{k}}{\cancel{k}c}$$

$$= \frac{\rho c^2}{c}$$

$$= \rho c. \quad \Rightarrow \quad \boxed{Z = \rho c}$$

$\boxtimes$

Problem 3 : Transmission & reflection of a transverse wave at a boundary between 2 media.

$\boxed{(a)}$ $\boxed{(i)}$ Incident wave

$A_i e^{i(k_1 x - \omega t)}$

Incident plane wave

$A_t e^{i(k_2 x - \omega t)}$

Transmitted plane wave

$A_r e^{i(-k_1 x - \omega t)}$

reflected plane wave

$k_1 \equiv$ wave # in string A

$k_2 \equiv$ wave # in string B.

1(b) Superposition principle yields:

Incident + reflected:

~~$y_i(x,t) = A_i e^{i(k_1 x - \omega t)} + A_r e^{ik_1 x}$~~

$$\boxed{y_A(x,t) = A_i e^{i(k_1 x - \omega t)} + A_r e^{i(-k_1 x - \omega t)}}$$

↑ resultant wave seen in string A.

$$\boxed{y_B(x,t) = y_{transmitted}(x,t) = A_t e^{i(k_2 x - \omega t)}}$$

↑ resultant wave seen in string B.

1(c) If A & B strings are joined together at $x = x_0$:

Boundary conditions (applied at $x = x_0$) are:

① $y_A(x_0, t) = y_B(x_0, t)$ ← 2 strings are joined to each other at $x = x_0$.

② $\left.\frac{\partial y_A}{\partial x}\right|_{x=x_0} = \left.\frac{\partial y_B}{\partial x}\right|_{x=x_0}$ ← No kinks. Smooth joint.

↑ At all t.

1(d) Pick $x_0 = 0$: Then above Boundary conditions become:

① $y_A(0,t) = y_B(0,t)$

$\Rightarrow \boxed{A_i + A_r = A_t}$

② $\left.\frac{\partial y_A(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial y_B(x,t)}{\partial x}\right|_{x=0}$

$\Rightarrow \boxed{ik_1 (A_i - A_r) = ik_2 A_t}$

so, we have:

$$\begin{cases} A_i + A_r = A_t & \text{:--- ①} \\ ik_1(A_i - A_r) = ik_2 A_t & \text{--- ②} \end{cases}$$

so:

$$ik_1(A_i - A_r) = ik_2(A_i + A_r)$$

$$\Rightarrow \quad A_i(ik_1 - ik_2) = A_r(ik_2 + ik_1)$$

$$\Rightarrow \quad \left|\frac{A_r}{A_i}\right|^2 = \left|\frac{ik_1 - ik_2}{ik_2 + ik_1}\right|^2$$

$$= \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

But $k_1 c_1 = \omega$ and $k_2 c_2 = \omega$.

$$= \left|\frac{\frac{\omega}{c_1} - \frac{\omega}{c_2}}{\frac{\omega}{c_1} + \frac{\omega}{c_2}}\right|^2$$

$$= \left|\frac{\frac{1}{c_1} - \frac{1}{c_2}}{\frac{1}{c_1} + \frac{1}{c_2}}\right|^2$$

And multiplying top & bottom by T:

$$= \left|\frac{\frac{T}{c_1} - \frac{T}{c_2}}{\frac{T}{c_1} + \frac{T}{c_2}}\right|^2$$

And $c_1 = \sqrt{\frac{T}{\rho_1}}$ $\quad c_2 = \sqrt{\frac{T}{\rho_2}}$

Tension same in both strings.

$$= \left|\frac{\frac{\rho_1 c_1^2}{c_1} - \frac{\rho_2 c_2^2}{c_2}}{\frac{\rho_1 c_1^2}{c_1} + \frac{\rho_2 c_2^2}{c_2}}\right|^2$$

$$= \boxed{\left|\frac{Z_A - Z_B}{Z_A + Z_B}\right|^2}$$

$\because Z_A = \rho_1 c_1$

$Z_B = \rho_2 c_2$.

As for the transmission coefficient $T = \left| \frac{A_t}{A_i} \right|^2$:

From the 2 Boundary conditions, we have:

$$A_i + A_r = A_t \quad \cdots \quad ①$$
$$ik_1 (A_i - A_r) = ik_2 A_t \quad \cdots \quad ②$$

$\Rightarrow \quad A_r = A_t - A_i$

so $\quad ik_1 (A_i - A_t + A_i) = ik_2 A_t$

$\Rightarrow \quad -ik_1 A_t = ik_2 A_t - 2ik_1 A_i$

$\Rightarrow \quad A_t(-ik_1 - ik_2) = -2ik_1 A_i$

$$\Rightarrow \quad \left| \frac{A_t}{A_i} \right|^2 = \left| \frac{2ik_1}{ik_1 + ik_2} \right|^2$$

$$= \left| \frac{2k_1}{k_1 + k_2} \right|^2$$

$$= \boxed{\left| \frac{2Z_A}{Z_A + Z_B} \right|^2}$$

← As before, when we computed R on previous page.

$\boxed{(e)}$ By taking the limit $Z_B \to \infty$: $\Rightarrow$ ~~[illegible]~~

$$T = \left| \frac{2Z_A}{Z_A + Z_B} \right|^2 \sim \left| \frac{Z_A}{Z_B} \right|^2$$
$$\sim \left| \frac{1}{Z_B} \right|^2$$
$$\to 0. \quad \text{As } Z_B \to \infty.$$

$\therefore$ $\boxed{T \to 0}$ (transmission coefficient.)

And: $R = \left| \frac{Z_A - Z_B}{Z_A + Z_B} \right|^2 \xrightarrow[Z_B \to \infty]{} \left| \frac{Z_B}{Z_B} \right|^2 = 1.$

$\therefore$ $\boxed{R = 1}$ (reflection coefficient) so, when $Z_B \to \infty$:

$$|A_i|^2 \cdot \underset{0}{T} = |A_t|^2$$

$\Rightarrow |A_t|^2 = 0$

$\Rightarrow \boxed{A_t = 0}$

$\Rightarrow$ No transmitted wave.

$\Rightarrow \boxed{y_B(x,t) = 0}$

$$|A_i|^2 \cdot \underset{1}{R} = |A_r|^2$$

$\Rightarrow \boxed{|A_i|^2 = |A_r|^2} \Rightarrow \boxed{A_r = A_i e^{i\delta}}$

where δ = phase

$\Rightarrow$ Incident wave: $\boxed{y_i(x,t) = A_i e^{i(k_1 x - \omega t)}}$

Reflected wave: $y_r(x,t) = A_r e^{i(-k_1 x - \omega t)}$

$$= \boxed{A_i e^{i(-k_1 x - \omega t + \delta)}}$$

· Can the phase shift δ be any arbitrary value?

Ans: No! To figure out δ, go back to the boundary conditions.

Note that $A_i + A_r = \cancel{A_t}$ 0

$\Rightarrow A_i = -A_r \Rightarrow A_i = -A_i e^{i\delta}$

$\Rightarrow e^{i\delta} = -1$

$\Rightarrow \boxed{\delta = \pi}$ in (radians.) (180°)

relative to incident wave, the phase shift.

∴ reflected wave is:

$$y_r(x,t) = A_i e^{i(-k_1 x - \omega t + \pi)} = \boxed{-A_i e^{i(-k_1 x - \omega t)}}$$

Physically, $Z_B \to \infty$ describes a situation in which

$$Z_B = \rho_2 C_2 = \rho_2 \sqrt{\frac{T}{\rho_2}} = \sqrt{\rho_2 T} \to \infty$$

Now, since tension T in both strings are the same, taking $T \to \infty$ would result in $Z_A \to \infty$ as well; which is not the case here.

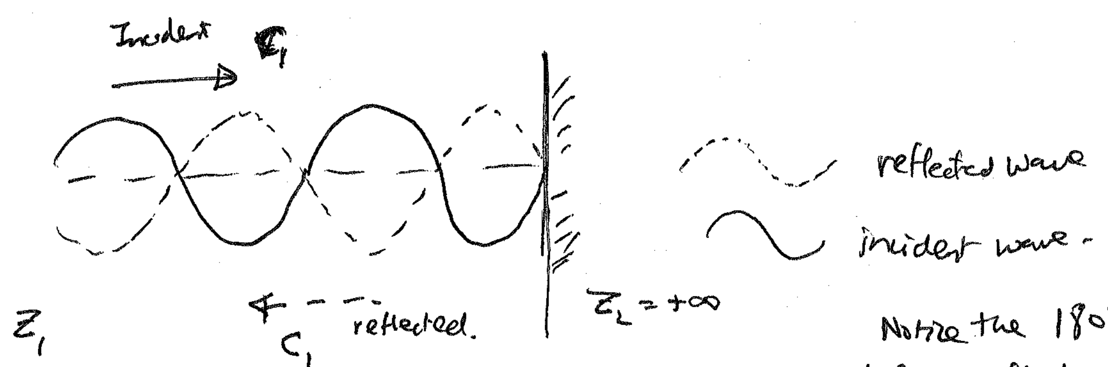
Hence $Z_B \to \infty$ in our case corresponds to $\rho_2 \to \infty$

(i.e. string B is very dense) compared to string A.

Notice that the resultant wave in string A (what you actually see with your eyes) is:

$$\begin{aligned} y_A(x,t) &= y_i(x,t) + y_r(x,t) \\ &= A_i e^{-i\omega t}\left[e^{ik_1x} - e^{-ik_1x}\right] \\ &= \boxed{2A_i i \sin(k_1 x)\, e^{-i\omega t}} . \end{aligned}$$

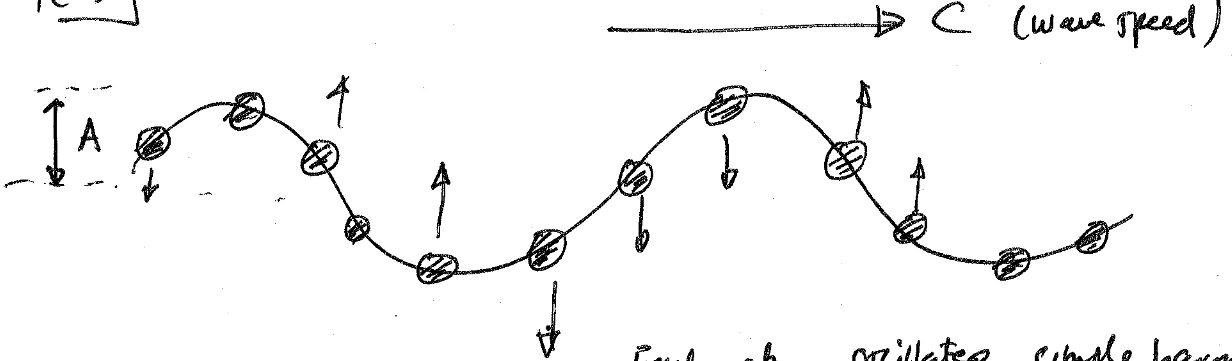
↑ standing wave with wavelength $\boxed{\lambda = \frac{2\pi}{k_1}}$



Notice the 180° phase shift upon reflection from the wall.

Problem 4 Transmission and reflection of energy

(a)



Each atom oscillates, simple harmonically, up and down.)

$y(x,t) = A\sin(kx - \omega t)$

while the wave itself travels with speed c horizontally.

describes wave. (with $kc = \omega$)

At a fixed value of x, (say $x = x_0$): $y(x_0, t) = A\sin(kx_0 - \omega t)$

describes a simple harmonic oscillator oscillating vertically up and down at $x = x_0$

Since kx_0 is constant over time, we can associate $kx_0 = \phi$ ← phase shift.

$$\Rightarrow \quad y(x_0, t) = A\sin(\phi - \omega t) \quad \leftarrow \text{Indeed, SHO.}$$

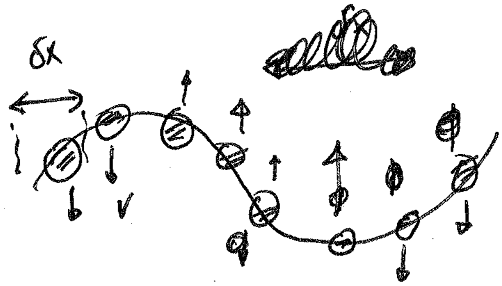
Notice that if (δm) is the mass of one "atom" drawn in the figure, its total energy is $E_{tot} = KE_{max}$ ← maximum KE that the SHO can have.

But
$$v = \frac{\partial y}{\partial t} = A\omega\cos(\phi - \omega t) \quad \Rightarrow \quad A\omega = V_{max} \text{ (max. speed of SHO at } x = x_0)$$

$$\Rightarrow \quad KE_{max} = \frac{1}{2}(\delta m)V_{max}^2$$

$$= \frac{(\delta m)}{2}(A\omega)^2$$

$$= \frac{(\rho\,\delta x)}{2}(A\omega)^2 \qquad \rho \equiv \text{uniform mass density.}$$



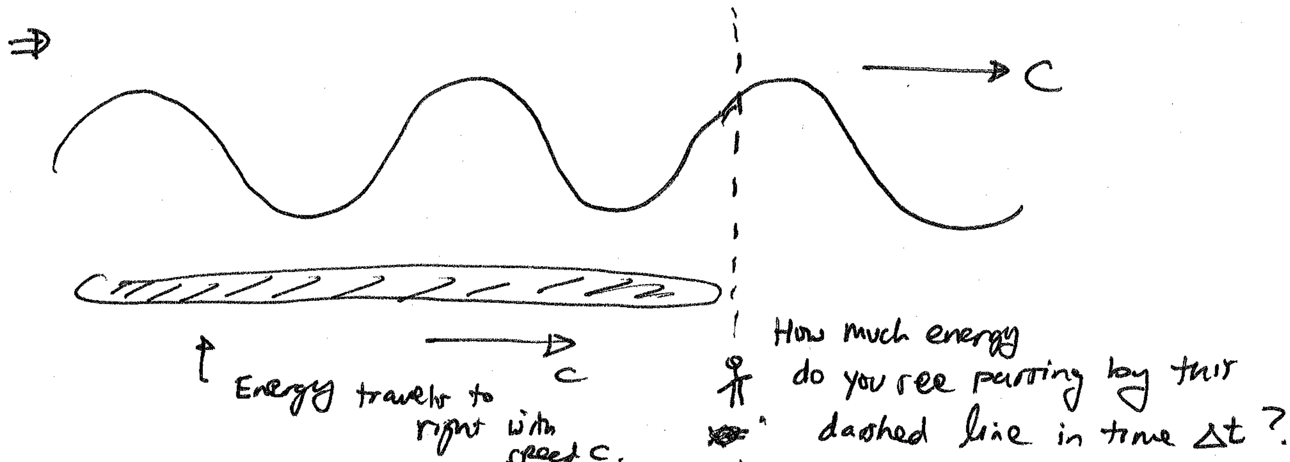
⇒ Energy contained within δx length of string is $\frac{(\rho\,\delta x)}{2}(A\omega)^2$.

$\therefore$ Energy density in wave $= \dfrac{\text{Energy contained within } \delta x \text{ length of string}}{\delta x}$

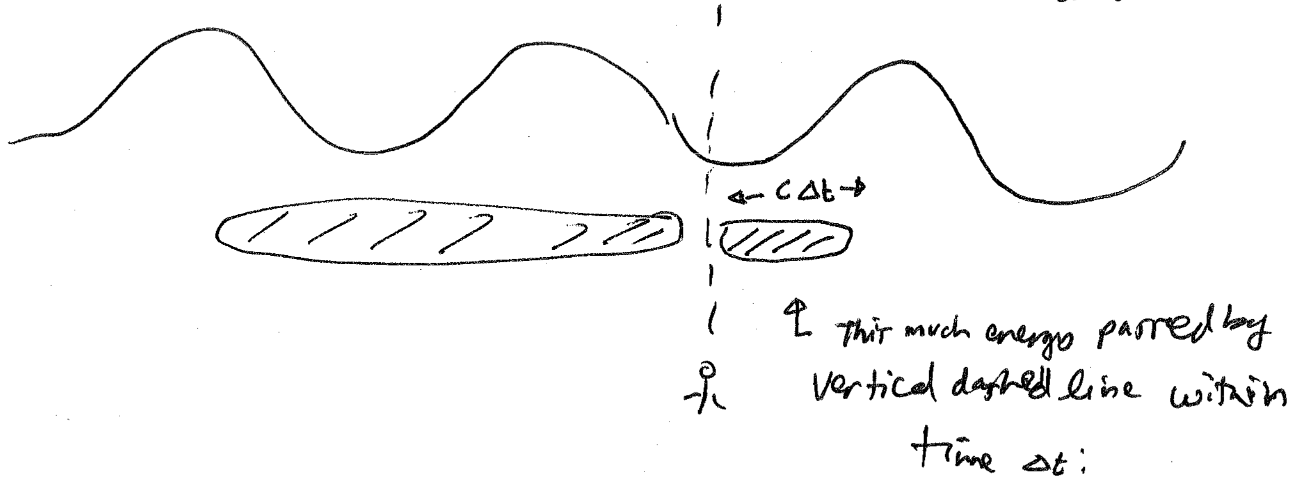
$$= \boxed{\frac{\rho (A\omega)^2}{2}}$$

1(b) Power = Energy delivered per unit time

$\Rightarrow$



Since $\rho_E = \dfrac{\rho A^2 \omega^2}{2}$ is energy contained per unit length of ~~string~~ wave,



$$\Rightarrow \quad \rho_E\, c(\Delta t) = \frac{\rho A^2 \omega^2}{2} c\,\Delta t = \text{total Energy passed by in time } \Delta t.$$

$$\Rightarrow \text{Power} = \frac{\text{Total energy passed by} \,\vdots\, \text{in time } \Delta t}{\Delta t}$$

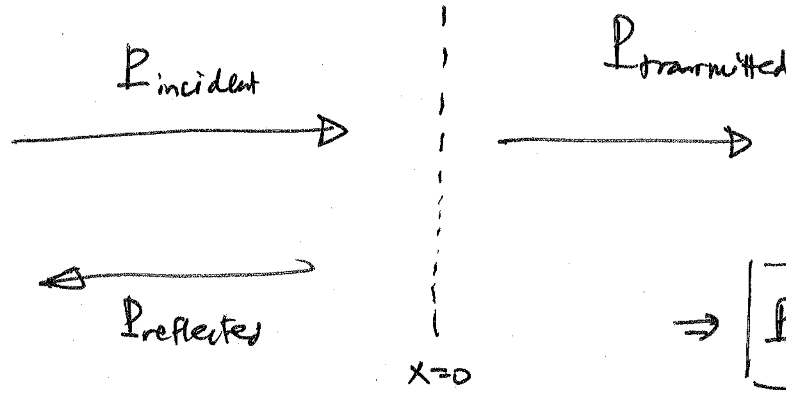
$$= \boxed{\frac{\rho A^2 \omega^2}{2} c} \leftarrow \text{Power:}$$

(c)

$$P = \frac{1}{2}\rho\omega^2 A^2 c.$$

$$= \frac{1}{2}\omega^2 A^2 \underbrace{(\rho c)}_{Z} = \boxed{\frac{1}{2} Z A^2 \omega^2.}$$

Now, by conservation of energy:



Want to prove this in this problem

$$\Rightarrow \boxed{P_{incident} = P_{transmitted} + P_{reflected}.}$$

i.e. (total Energy arriving at $x=0$)

= (total energy leaving $x=0$) (otherwise, energy "piles up" at $x=0$.)

← NOT physical.

$P_{incident} = \frac{1}{2} Z_A A_i^2 \omega^2.$ $\quad P_{transmitted} = \frac{1}{2} Z_B A_t^2 \omega^2$

$P_{reflected} = \frac{1}{2} Z_A A_r^2 \omega^2$

Notice that: $\quad P_{reflected} + P_{transmitted} = \frac{1}{2}\omega^2 \left[Z_A A_r^2 + Z_B A_t^2 \right]$

~~Note~~ Furthermore, note that $\quad R = \left|\frac{A_r}{A_i}\right|^2 = \frac{A_r^2}{A_i^2}$ ← (No need for modulus symbol $|\cdots|$ since A_r and A_i are real #'s.)

$$T = \frac{A_t^2}{A_i^2}$$

$$\therefore P_{reflected} + P_{transmitted} = \frac{1}{2}\omega^2 A_i^2 \left[Z_A R + Z_B T \right]$$

over →

Continued from previous pg :

$$P_{reflected} + P_{transmitted} = \frac{1}{2}\omega^2 A_i^2 \left[Z_A R + Z_B T \right]$$

$$= \frac{1}{2}\omega^2 A_i^2 \left[Z_A \left(\frac{Z_A^2 - 2Z_A Z_B + Z_B^2}{(Z_A+Z_B)^2} \right) + Z_B \left(\frac{4Z_A^2}{(Z_A+Z_B)^2} \right) \right]$$

$$= \frac{\omega^2 A_i^2}{2} \left[\frac{Z_A^3 + 2Z_A^2 Z_B + Z_A Z_B^2}{(Z_A+Z_B)^2} \right]$$

$$= \frac{\omega^2 A_i^2 Z_A}{2} \left[\frac{Z_A^2 + 2Z_A Z_B + Z_B^2}{(Z_A+Z_B)^2} \right]$$

$$= \frac{\omega^2 A_i^2 Z_A}{2} \underbrace{\frac{(Z_A+Z_B)^2}{(Z_A+Z_B)^2}}_{"1}$$

$$= \frac{Z_A A_i^2 \omega^2}{2}$$

$$= P_{incident}$$

$\therefore$ Indeed, we have just proved that $\boxed{P_{incident} = P_{reflected} + P_{transmitted}}$

1(d) $\boxed{\frac{P_{reflected}}{P_{incident}} = \frac{\frac{1}{2}\omega^2 A_i^2 Z_A R}{\frac{1}{2} Z_A A_i^2 \omega^2}} = \boxed{R = \left(\frac{Z_A - Z_B}{Z_A + Z_B} \right)^2}$

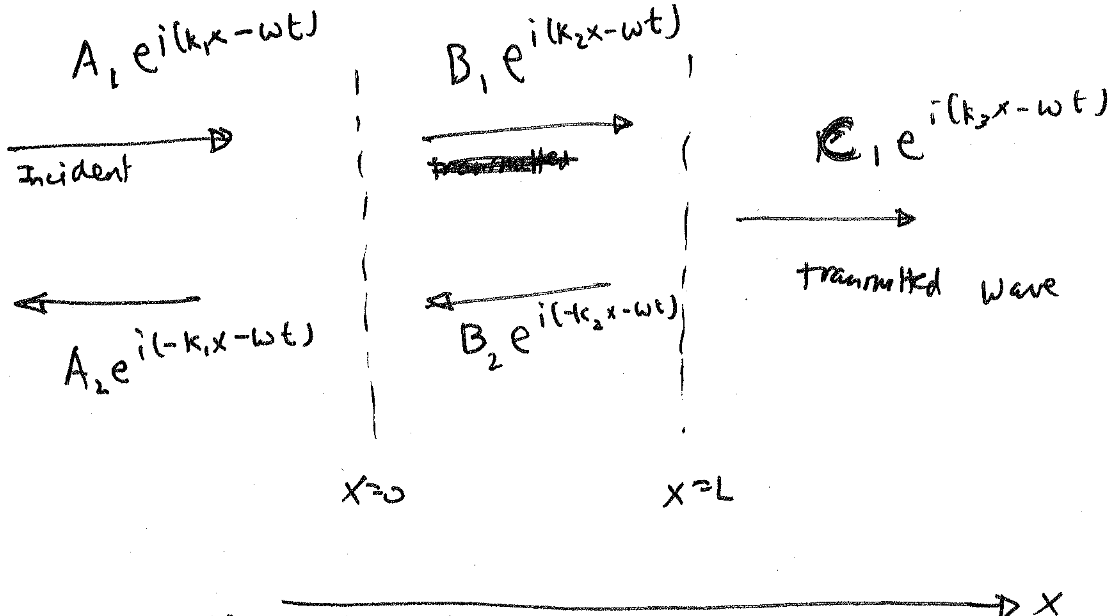
↑ As found in Problem 3.(d).

$$\frac{P_{transmitted}}{P_{incident}} = \frac{\frac{1}{2}\omega^2 A_i^2 Z_B T}{\frac{1}{2}\omega^2 A_i^2 Z_A} = \frac{Z_B}{Z_A}\left(\frac{2Z_A}{Z_A+Z_B} \right)^2$$ ← As found in Problem 3 (d).

$$= \boxed{\frac{4Z_A Z_B}{(Z_A+Z_B)^2}}$$

Problem 5 : Anti-glare coating on lenses (Impedance matching)

1(a)



$\Rightarrow$ $y_A(x,t) = A_1 e^{i(k_1x-\omega t)} + A_2 e^{i(-k_1x-\omega t)}$ $\quad$ $y_B(x,t) = B_1 e^{i(k_2x-\omega t)} + B_2 e^{i(-k_2x-\omega t)}$

$y_C(x,t) = C_1 e^{i(k_3x-\omega t)}$

1(b) At $x=0$: Boundary condition (BC)

BC1 : $y_A(0,t) = y_B(0,t)$ $\quad$ ← 2 strings (A & B) are joined to each other at $x=0$.

$\Rightarrow$ $A_1 + A_2 = B_1 + B_2$ $\quad$ .... $\boxed{\text{Eq'n}(1)}$

BC2 : $\left.\dfrac{\partial y_A}{\partial x}\right|_{x=0} = \left.\dfrac{\partial y_B}{\partial x}\right|_{x=0}$ $\quad$ ← No kinks. (smoothly varying string)

$\Rightarrow$ $ik_1A_1 - ik_1A_2 = ik_2(B_1 - B_2)$

$\Rightarrow$ $k_1(A_1 - A_2) = k_2(B_1 - B_2)$ $\quad$ .... $\boxed{\text{Eq'n}(2)}$

Similarly, boundary conditions at $x = L$ are:

BC3: $y_B(x=L,t) = y_c(x=L,t)$

$$\Rightarrow \boxed{B_1 e^{ik_2 L} + B_2 e^{-ik_2 L} = C_1 e^{ik_3 L}} \quad \ldots \boxed{\text{eqn (3)}}$$

BC4: $\left.\frac{\partial y_B}{\partial x}\right|_{x=L} = \left.\frac{\partial y_c}{\partial x}\right|_{x=L}$

$$\Rightarrow ik_2 [B_1 e^{ik_2 L} - B_2 e^{-ik_2 L}] = ik_3 C_1 e^{ik_3 L}$$

$$\Rightarrow \boxed{k_2 [B_1 e^{ik_2 L} - B_2 e^{-ik_2 L}] = k_3 C_1 e^{ik_3 L}} \quad \ldots \boxed{\text{eqn (4)}}$$

$\boxtimes$

(c) $$\frac{P_{\text{transmitted}}}{P_{\text{incident}}} = \frac{\frac{1}{2} Z_c \omega^2 |C_1|^2}{\frac{1}{2} Z_A \omega^2 |A_1|^2}$$

From (c) in problem 4.

$$= \frac{Z_c}{Z_A} \left|\frac{C_1}{A_1}\right|^2$$

← Modulus squared (since in this particular problem, A, B, and C can be complex-valued amplitudes.

Now, figure out $\left|\frac{C_1}{A_1}\right|^2$ using the eqns (1), (2), (3), and (4)

we found (4 boundary condition) in ~~[illegible]~~ on pg 19 and pg 20.

First : Eqn (3) + $\frac{1}{k_2}$ · Eqn (4) gives:

$$2B_1 e^{ik_2L} = C_1 e^{ik_3L} + \frac{k_3}{k_2} C_1 e^{ik_3L}$$

$$= C_1 e^{ik_3L}[1 + k_3/k_2]$$

$$\Rightarrow \quad B_1 = \frac{C_1 e^{i(k_3-k_2)L}}{2}[1 + k_3/k_2]$$

Next, Eqn (3) $-$ $\frac{1}{k_2}$ Eqn (4) yields :

$$2B_2 e^{-ik_2L} = C_1 e^{ik_3L} - \frac{k_3}{k_2} e^{ik_3L} C_1$$

$$\Rightarrow \quad B_2 = \frac{C_1 e^{i(k_3L + k_2L)}}{2}[1 - k_3/k_2]$$

(so far, we've expressed B_1 & B_2 in terms of C_1).

Next, Eqn (1) + $\frac{1}{k_1}$ · Eqn (2) yields :

$$2A_1 = B_1 + B_2 + \frac{k_2}{k_1}(B_1 - B_2)$$

$$= B_1\left[1 + \frac{k_2}{k_1}\right] + B_2\left[1 - \frac{k_2}{k_1}\right]$$

Express B_1 & B_2 in terms of C_1, then we can take C_1/A_1, which is what we want.

$$\Rightarrow \quad 2A_1 = \frac{C_1}{2} e^{i(k_3-k_2)L} \left[1+\frac{k_3}{k_2}\right]\left[1+\frac{k_2}{k_1}\right] + \frac{C_1 e^{i(k_3+k_2)L}}{2} \left[1-\frac{k_3}{k_2}\right]\left[1-\frac{k_2}{k_1}\right]$$

<u>Notation change</u>: $k_1 \equiv k_A \qquad k_2 \equiv k_B, \qquad k_3 \equiv k_c.$

Notice that $\dfrac{k_c}{k_B} = \dfrac{\cancel{\omega}}{c_3}\dfrac{c_2}{\cancel{\omega}} \leftarrow \because \begin{cases} k_1 c_1 = \omega \\ k_2 c_2 = \omega \\ k_3 c_3 = \omega. \end{cases}$

$$= \frac{T c_2}{T c_3} \qquad \text{since } c_j = \sqrt{\frac{T}{\rho_j}}$$

$$= \frac{\rho_3 c_3^{\cancel{2}} \cancel{c_2}}{\rho_2 c_2^{\cancel{2}} \cancel{c_3}} \qquad \Rightarrow \rho_j c_j^2 = T$$

$$= \frac{\rho_3 c_3}{\rho_2 c_2} = \frac{Z_c}{Z_B} \qquad \text{In fact, } \boxed{\frac{Z_n}{Z_m} = \frac{k_n}{k_m}}$$

<u>By definition</u> $\rightarrow \equiv r_{CB}$ $\qquad\qquad \boxed{} \equiv r_{nm}.$

Thus, the eqn at the top of this page becomes :

$$\frac{A_1}{C_1} = \frac{1}{4}\left\{ e^{i(k_c-k_B)L} [1+r_{CB}][1+r_{BA}] + e^{i(k_B+k_c)L} [1-r_{CB}][1-r_{BA}] \right\}$$

$$= \frac{e^{ik_cL}}{4}\left\{ e^{-ik_BL} [1+r_{CB}][1+r_{BA}] + e^{ik_BL} [1-r_{CB}][1-r_{BA}] \right\}$$

$$= \frac{e^{ik_cL}}{4}\left\{ \begin{array}{l} (e^{ik_BL} + e^{-ik_BL}) + r_{BA}(e^{-ik_BL} - e^{ik_BL}) \\ + r_{CB}(e^{-ik_BL} - e^{ik_BL}) + r_{CB} r_{BA} [e^{-ik_BL} + e^{ik_BL}] \end{array} \right\}$$

over →

$$\Rightarrow \quad \frac{A_1}{C_1} = \frac{e^{ik_c L}}{4} \left\{ 2\cos(k_B L) + r_{BA}\left(-2i\sin(k_B L)\right) - 2i\sin(k_B L) r_{CB} + r_{CB} r_{BA}\, 2\cos(k_B L) \right\}$$

$$= \frac{e^{ik_c L}}{2} \left\{ \cos(k_B L)\left[1 + r_{CB} r_{BA}\right] - i\sin(k_B L)\left[r_{BA} + r_{CB}\right] \right\}$$

$$\Rightarrow \quad \left|\frac{C_1}{A_1}\right|^2 = \frac{4}{\cos^2(k_B L)\left[1 + \underbrace{r_{BA} r_{CB}}_{= r_{CA}}\right]^2 + \sin^2(k_B L)\left[r_{BA} + r_{CB}\right]^2}$$

modulus squared

(Notice $|e^{ik_c L}| = 1$.

$\therefore$ From pg 20 : $$\frac{P_{\text{transmitted}}}{P_{\text{incident}}} = \overbrace{\left(\frac{Z_C}{Z_A}\right)}^{r_{CA}} \left|\frac{C_1}{A_1}\right|^2$$

$$= \frac{4 r_{CA}}{\cos^2(k_B L)\left[1 + r_{CA}\right]^2 + \sin^2(k_B L)\left[r_{BA} + r_{CB}\right]^2}$$

$$= \frac{\overbrace{4 r_{CA} \cdot r_{AC}^2}^{= r_{AC}}}{\cos^2(k_B L)\left\{(1 + r_{CA})(r_{AC})\right\}^2 + \sin^2(k_B L)\left\{(r_{BA} + r_{CB}) r_{AC}\right\}^2}$$

$$= \boxed{\frac{4 r_{AC}}{(1 + r_{AC})^2 \cos^2(k_B L) + (r_{AB} + r_{BC})^2 \sin^2(k_B L)}}$$

(d) Let $L = \frac{\lambda_B}{4} = \frac{2\pi}{4k_B} = \frac{\pi}{2k_B}$

$\lambda_B = \frac{2\pi}{k_B}$

and $Z_B = \sqrt{Z_A Z_C}$

Then:

Plugging into: $\frac{P_{\text{transmitted in C}}}{P_{\text{incident in A}}} = \frac{4 r_{AC}}{(1+r_{AC})^2 \underbrace{\cos^2(k_B L)}_{\cos^2\left(\frac{\pi}{2}\right) = 0} + (r_{AB}+r_{BC})^2 \underbrace{\sin^2(k_B L)}_{\sin^2\left(\frac{\pi}{2}\right) = 1}}$

$$= \frac{4 r_{AC}}{(r_{AB}+r_{BC})^2}$$

$\because Z_B = \sqrt{Z_A Z_C}$

$$\frac{r_{BC}}{r_{AB}} = \frac{Z_B}{Z_C}\frac{Z_B}{Z_A} = \frac{Z_A Z_C}{Z_A Z_C} = 1.$$

$$= 4\frac{r_{AC}}{(r_{AB})^2}\frac{1}{\left[1+\frac{r_{BC}}{r_{AB}}\right]^2}$$

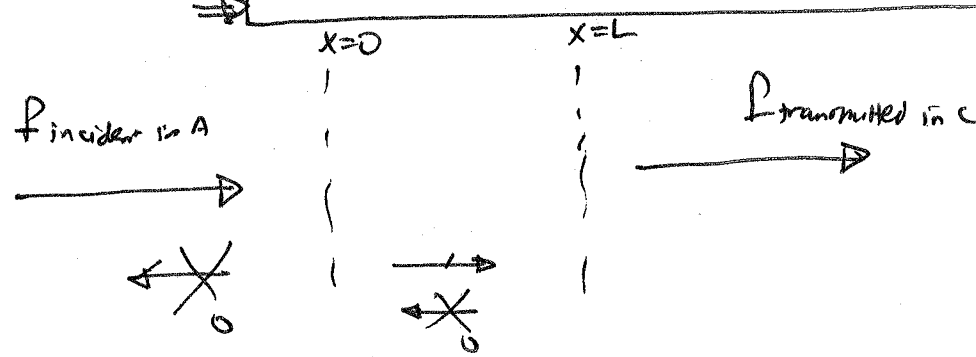
$$\frac{r_{AC}}{(r_{AB})^2} = \frac{Z_A}{Z_C}\frac{Z_B^2}{Z_A^2} = \frac{Z_A Z_C}{Z_A Z_C} = 1.$$

$$= \frac{r_{AC}}{(r_{AB})^2}$$

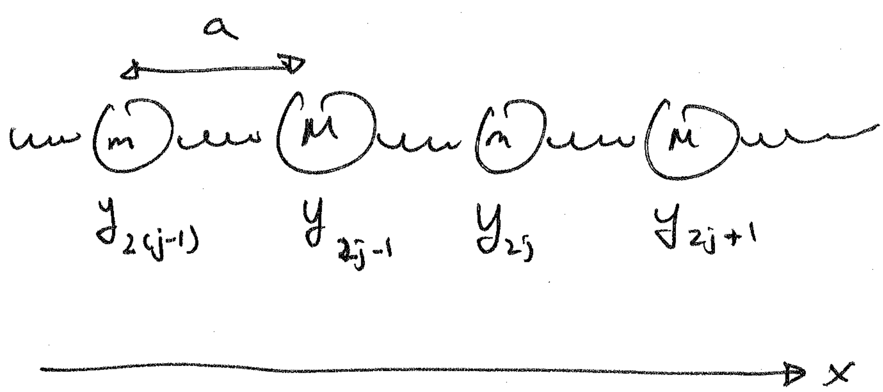
$$= 1.$$

$\therefore$ $\boxed{P_{\text{transmitted in C}} = P_{\text{incident in A}}.}$

$\Rightarrow$ All of energy transmitted through the middle slab.



Problem 6: 1D crystal made up of 2 kinds of ions



(a) On 2j-th ion:

$$m\ddot{y}_{2j} = -k\left(y_{2j} - y_{2j-1}\right) + k\left(y_{2j+1} - y_{2j}\right)$$

$$= k\left[y_{2j+1} + y_{2j-1} - 2y_{2j}\right]$$

$$= \boxed{\frac{(ka)}{a}\left[y_{2j+1} + y_{2j-1} - 2y_{2j}\right]}$$

where $\boxed{ka \equiv T}$ ← defined on the Pset handout

Similarly, it's easy to see that:

$$\boxed{M\frac{d^2 y_{2j+1}}{dt^2} = \frac{T}{a}\left[y_{2j+2} + y_{2j} - 2y_{2j+1}\right]}$$

for the $(2j+1)$st ion. M.

☒

(b) $\mathbb{C}$- equivalent EOMs:

$$\begin{cases} m\ddot{\tilde{y}}_{2j} = \frac{T}{a}\left(\tilde{y}_{2j+1} + \tilde{y}_{2j-1} - 2\tilde{y}_{2j}\right) \\ M\ddot{\tilde{y}}_{2j+1} = \frac{T}{a}\left(\tilde{y}_{2j+2} + \tilde{y}_{2j} - 2\tilde{y}_{2j+1}\right) \end{cases}$$

$\tilde{y}_i = \mathbb{C}$-valued.

Guess Normal modes to be:

$$\begin{cases} \tilde{y}_{2j} = A_m e^{i(\omega t - 2jka)} \\ \tilde{y}_{2j+1} = A_M e^{i(\omega t - (2j+1)ka)} \end{cases}$$

Plugging into above EoMs yields:

$$-\omega^2 A_m e^{-2ijka} = \frac{T}{ma}\left[A_M e^{-i(2j+1)ka} + A_M e^{-i(2j-1)ka} - 2A_m e^{-i2jka}\right]$$

$$\Rightarrow -\omega^2 A_m = \frac{T}{ma}\left[A_M e^{-ika} + A_M e^{ika} - 2A_m\right]$$

$$\Rightarrow -\omega^2 A_m = \frac{T}{ma}\left[2A_M\cos(ka) - 2A_m\right] \quad \text{.... eqn (1)}$$

And:

$$-\omega^2 A_M e^{-i(2j+1)ka} = \frac{T}{Ma}\left[A_m e^{-i2(j+1)ka} + A_m e^{-2ijka} - 2A_M e^{-i(2j+1)ka}\right]$$

$$\Rightarrow -\omega^2 A_M e^{-ika} = \frac{T}{Ma}\left[A_m e^{-i2ka} + A_m - 2A_M e^{-ika}\right]$$

$$\Rightarrow -\omega^2 A_M = \frac{T}{Ma}\left[A_m e^{-ika} + A_m e^{ika} - 2A_M\right]$$

$$\Rightarrow -\omega^2 A_M = \frac{T}{Ma}\left[2A_m\cos(ka) - 2A_M\right] \quad \text{--- eqn (2)}$$

Hence:

Eqn (1): $-\omega^2 A_m = \frac{2T}{ma}\left[A_M \cos(ka) - A_m\right]$

Eqn (2): $-\omega^2 A_M = \frac{2T}{Ma}\left[A_m \cos(ka) - A_M\right]$

↳ 2 eqns with 2 "unknowns": A_m & A_M.

Solving gives:
(just some algebra)

$$\boxed{\omega^2 = \frac{T}{a}\left(\frac{1}{m}+\frac{1}{M}\right) \pm \frac{T}{a}\left\{\left(\frac{1}{m}+\frac{1}{M}\right)^2 - \frac{4\sin^2(ka)}{mM}\right\}^{1/2}}$$

↳ dispersion relationship

First, look at end values of k:

[c] • When $k=0$: $\omega^2 = \frac{2T}{a}\left(\frac{1}{m}+\frac{1}{M}\right)$.

• when $k = \frac{\pi}{2a}$: $\omega^2 = \frac{2T}{Ma}$

Also, look at the limiting cases: when k small: (i.e. $|ka| \ll 1$)

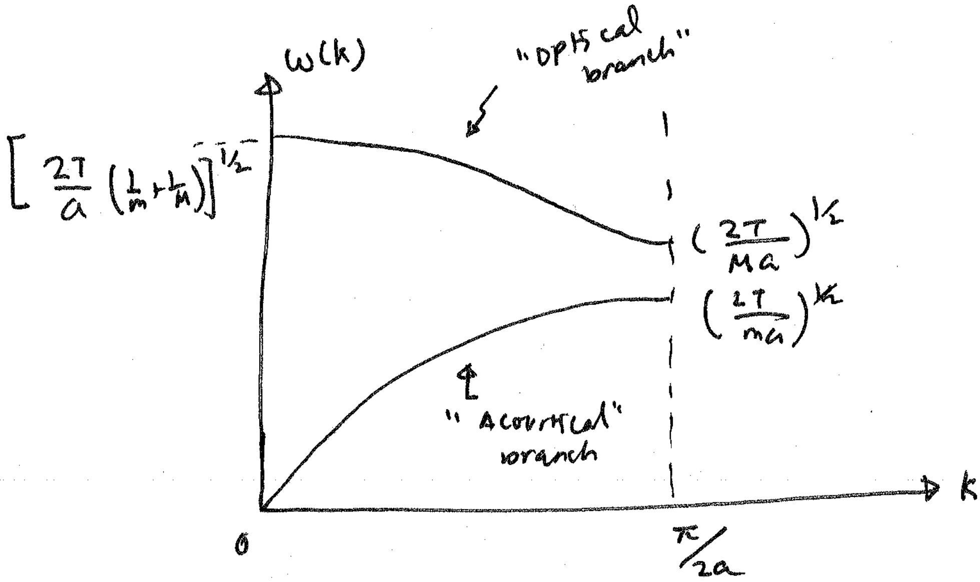
$\sin^2(ka) \approx (ka)^2$

$\Rightarrow$ For the $\ominus$ sign root ("lower branch"):

$$\boxed{\omega_-^2 \approx \frac{2T(ka)^2}{a(M+m)}}$$ (when k small)

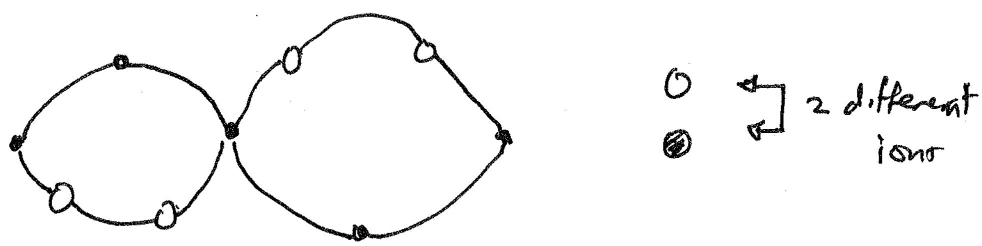
With these information, we plot $\omega(k)$ vs k.

→ over



$(m > M)$.

$\boxed{1(d)}$ **optical mode:**



Acoustical mode: