Calculus II (MITES 09)



# Double and Triple integrals, and flux through a flat sheet

**e.g. 1** $\displaystyle\int_{x=0}^{3}\int_{y=2}^{5}(x^2+xy)\,dx\,dy$

We can do this in 2 ways: ① Do x-integral, then y-integral.
② Do y-integral, then x-integral.
Both should yield the same answer.

① : $\displaystyle\int_{y=2}^{5}\int_{x=0}^{3}(x^2+xy)\,dx\,dy = \int_{y=2}^{5} dy\left\{\frac{x^3}{3}+\frac{x^2}{2}y\right\}\Bigg|_{x=0}^{3}$

$\displaystyle = \int_{y=2}^{5} dy\left\{\frac{27}{3}+\frac{9y}{2}\right\}$

$\displaystyle = \left[\frac{27}{3}y+\frac{9y^2}{4}\right]\Bigg|_{y=2}^{5}$

$\displaystyle = \frac{135}{3}+\frac{225}{4}-\frac{54}{3}-\frac{36}{4}$

$\displaystyle = \frac{81}{3}+\frac{189}{4} = \boxed{27+\frac{189}{4}}$

② $\displaystyle\int_{x=0}^{3}\int_{y=2}^{5}(x^2+xy)\,dx\,dy = \int_{x=0}^{3} dx\left\{x^2y+\frac{xy^2}{2}\right\}\Bigg|_{y=2}^{5}$

$\displaystyle = \int_{x=0}^{3} dx\left\{5x^2+\frac{25x}{2}-2x^2-2x\right\}$

$\displaystyle = \left[\frac{5x^3}{3}+\frac{25x^2}{4}-\frac{2x^3}{3}-x^2\right]\Bigg|_{x=0}^{3}$

$\displaystyle = 45+\frac{225}{4}-18-9$

$\displaystyle = \boxed{27+\frac{189}{4}}$

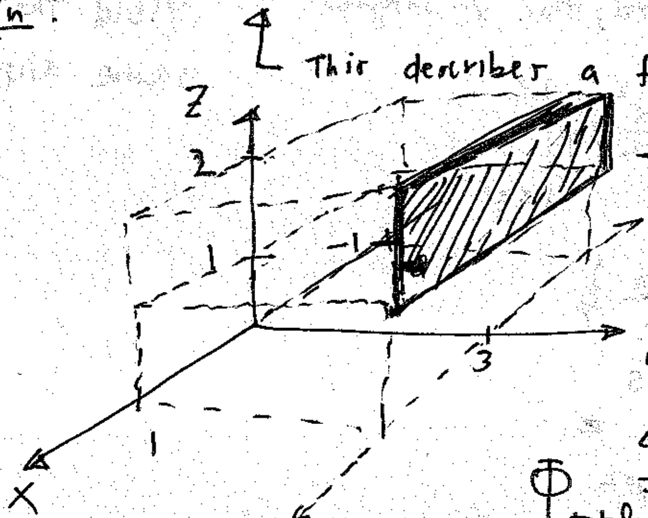
Same ✓

## e-2.2 Calculating flux through a surface:

Consider a vector field $\vec{v}(x,y,z) = (3xy,\ y^2,\ xz + z^2)$.
Compute the flux through a rectangular sheet defined by the set of points $\{(x,y,z) : -1 \le x \le 1,\ y=3,\ 1 \le z \le 2\}$

**Sol'n:**

This describes a flat sheet parallel to xz-plane




shaded rectangle (want to compute flux through this sheet.)

total flux

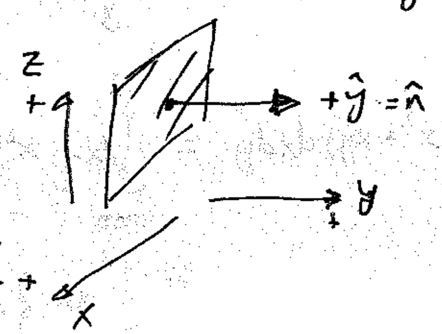
$$\Phi_{total} = \int_{▨} (\vec{v} \cdot \hat{n})\, da$$

unit normal vector $= \hat{y}$

$\vec{v} \cdot \hat{n} = \vec{v} \cdot \hat{y}$

$= (3xy,\ y^2,\ xz + z^2) \cdot (0,1,0)$

$= y^2 = 3^2 = \boxed{9}$ since $y = 3$ on the flat sheet




$da = dx\,dz$ ;

so: $\Phi_{total} = \int_{▨} dx\,dz\ 9$

$= \int_{x=-1}^{1} \int_{z=1}^{2} 9\, dx\, dz$

$= \int_{x=-1}^{1} \left[ 9z \right]_{z=1}^{2} dx$

$= \int_{x=-1}^{1} [9]\, dx$

$= 9x \Big|_{x=-1}^{1}$

$= \boxed{18}$

$\therefore\ \boxed{\Phi_{total} = 18}$

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