MITES 2009 : Calculus II - Practice Final Exam.

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Problem 1. Vectors:

(a.) Find a vector that is parallel to the line described by $(x, y, z) = (2, 1, -5) + t(16, 2, 9)$.

(b.) Find a vector that is orthogonal (i.e., perpendicular) to the plane $2x - 10y + 5z = a^2$, where a is some arbitrary constant.

(c.) Find a vector that makes an angle of 60^o to $\hat{y}$ and makes equal angles with $\hat{x}$ and $\hat{z}$.

Problem 2. Equation of a line:

(a.) Find the equation of the line that passes through $(-3, 9, 2)$ and is parallel to the $x - axis$.

(b.) Find the equation of the line passing through the two points $(0, -1, 2)$ and $(\pi, -\pi, 8)$.

(c.) Find the equation of the line passing through the point $(-2, \pi, 3)$ that intersects and is perpendicular to the line $(x, y, z) = (-1, -2, -1) + t(1, 2, 5)$.

Problem 3. Equation of a plane:

(a.) Find the equation for the plane perpendicular to $(-1, 1, -1)$ and passing through $(1, 1, 1)$.

(b.) Find the equation for the plane perpendicular to the line described by $(x, y, z) = (1, 1, 1) + t(-2, 1, 2)$ and passing through $(-1, 1, 3)$.

(c.) Consider the plane described by the equation $3x + 2y - \pi z - 2\pi = 0$. Find at least three points that lie on this plane. Write down two vectors that are normal to this plane.

Problem 4. Orthogonal projection of a vector onto another vector:

(a.) Find the orthogonal projection of the vector $(1, 1, 0)$ onto the vector $(1, -2, 0)$. What is the length of this projected vector?

(b.) Show that the distance from the point (x_1, y_1) to the line $ax + by = c$ is $\dfrac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$.

Problem 5. Gradients and directional derivatives:

You are hiking on a mountain whose altitude is $z(x, y) = -x^3 + e^{xy} + 3sin(x + y^2)$. Your position as a function of time t is $\vec{c}(t) = (sin(t) + t, t^2 + e^t)$. So $x(t) = sin(t) + t$ and $y(t) = t^2 + e^t$ are the longitude and latitude of your position at time t respectively.

(a.) What is your altitude at time t?

(b.) Using your answer to (a), calculate the instantaneous rate of change in your altitude with respect to time, at $t = 2\pi$.

Next, let's compute the same quantity as in (b) (i.e, $\frac{dz(\vec{c}(t))}{dt}$) but using directional derivatives computed along the trajectory $\vec{c}(t)$. To do this,

(c.) First, compute the gradient of z at position (x, y). (i.e., $\nabla z(x, y)$).

(d.) Next, compute your instantaneous velocity at time t on the trail. (i.e., $\frac{d\vec{c}(t)}{dt}$).

(e.) Finally, compute $\frac{dz(\vec{c}(t))}{dt}$ using the two quantities you calculated above.

(f.) Find the equation of a plane tangent to the graph of $z(x, y)$ at $(x, y) = (0, 2)$.

(g) Write down any vector that is normal (i.e., perpendicular) to the plane you found in (f).

Problem 6. Ordinary Differential Equations:

Solve the following ordinary differential equations. Be sure to write down the most general solution of the equation which contains the right number of arbitrary constants. Here, y = y(x).

(a) $y'' + y' + y = 3x$

(b) $y'' + 3y = sin(x)$

(c) $y' + ay = 0$. After writing down the general solution, write down the solution if y(0) = 0. a is a constant.

(d) $y' = g - ay^2$, where g and a are positive constants.

(e) $2y''' + y'' - 5y' + 2y = 0$

(f) $y'' - 5y' + 6y = 0$.

(g) $y'' + a^2y = 3sin(10x) + 100sin(3x) + 2cos(9x) + 27cos(74x)$

(h) $y'' + c^2y = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$

Problem 7. Double and triple integrals:

Evaluate the following double integrals.

(a) $\int_{y=0}^{1} \int_{x=0}^{1} (x^3 + y^2)dxdy$

(b) $\int_{y=0}^{1} \int_{x=0}^{1} ye^{xy}dxdy$

(c) $\int_{y=0}^{1} \int_{x=0}^{1} (xy)^2 cos(x^3) dxdy$

(d) $\int_{y=0}^{1} \int_{x=0}^{1} (x^2 + 2xy + y\sqrt{x}) dxdy$

(e) $\int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} x^2 dxdydz$

(f) $\int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} e^{-xy} dxdydz$

(g) $\int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=x^2+y^2}^{x+y} dzdydx$

Problem 8. Flux and Divergences:

(a) Compute the divergence of the vector field $\vec{v}(x, y) = (-y, x)$. (i.e., what is $\nabla \cdot \vec{v}$)?

(b) Compute the divergence of the vector field $\vec{v}(x, y, z) = (e^{xy}, -e^{xy}, e^{yz})$.

(c) Compute the divergence of the vector field $\vec{v}(x, y, z) = (yz, xz, xy)$.

(d) Compute the divergence of the vector field $\vec{v}(x, y, z) = (x, y + cos(x), z + e^{xy})$.

(e) Compute the divergence of the vector field $\vec{v}(x, y, z) = (x^2, (x + y)^2, (x + y + z)^2)$.

(f) Write down any vector field $\vec{v}(x, y)$ whose divergence is zero and resides on the xy-plane (i.e. z-component of $\vec{v}(x, y)$ is zero).

(g) Write down any vector field $\vec{v}(x, y)$ whose divergence is zero and resides on the xz-plane (i.e. y-component of $\vec{v}(x, y)$ is zero).

(h) Consider a force field $\vec{F}(x, y, z) = (2x, 5y, 3z)$. Compute the flux of this force field through a rectangle described by the set of points $\{(x, y, z)|2 \le x \le 5, y = 0, -5 \le z \le 5\}$.

(i) Consider a fluid whose velocity field is $\vec{v}(x, y, z) = (x - y, xyz, e^{x+z^2})$. Compute the flux of this fluid field through a rectangle described by the set of points $\{(x, y, z)|2 \le x \le 5, 2 \le y \le \pi, z = 10\}$.

(j) Consider a field of cosmic rays described by $\vec{R}(x, y, z) = (x - y + sin(z), xz + xyz, x^3 + 2z + 3y)$. Compute the flux of this fluid field through a rectangle described by the set of points $\{(x, y, z)|2 \le x \le 5, y = \pi, 0 \le z \le 5\}$.

Problem 9. Fourier Series:

Find the Fourier series representation for the following functions on the x-interval $[-\pi, \pi]$.

(a) $f(x) = x$

(b) $g(x) = 1$

(c) $h(x) = x^2$

(d) $k(x) = 34sin(2x) + 100sin(100x) + 2 + 45cos(23x) + 100sin(131x) + cos(2x)$

(e) $L(x) = cos(2x)cos(4x) - sin(2x)sin(4x) + cos(100x)sin(x) + sin(100x)cos(x)$.