Lecture 1 : Calculus II (MITES 09) Hyun Y.

(Mon) June 22, 09

- Functions

Def^n : A function is a rule which assigns, to each of certain real numbers, some other single real number.

Ex1 : The rule which assigns to each number the square of that number

(i.e. $x \xrightarrow[\text{"f"}]{\text{rule}} x^2$: can write this as $f(x) = x^2$)

"f takes "x" and turns it into x^2".

Ex2 : The rule which assigns to each number y the number $\dfrac{y^3+3y+5}{y^2+1}$

(i.e. $y \xrightarrow[\text{"g"}]{\text{rule}} \dfrac{y^3+3y+5}{y^2+1}$: Can write this as $g(y) = \dfrac{y^3+3y+5}{y^2+1}$)

"g takes "y" and turns it into $\dfrac{y^3+3y+5}{y^2+1}$".

Ex3 : The rule which assigns to each number x satisfying $-17 \le x \le \pi/3$ the number x^2.

(i.e. If $-17 \le x \le \pi/3$ then $x \xrightarrow[\text{rule}]{\text{"h"}} x^2$: Can write this as $h(x) = x^2$, if $-17 \le x \le \pi/3$.)

Notice that in this example, nothing is said about what the "rule" is if x does not meet the condition $-17 \le x \le \pi/3$.

This does not mean that h assigns zero for $x > \pi/3$ and $x < -17$.

It simply means that we do not know what to do when x is not part of the interval $[-17, \pi/3]$.

Here, we say that the domain of function h is : $-17 \le x \le \pi/3$

Def^n : The set of numbers to which a function does apply is called the domain of the function.

Ex 4. The rule which assigns to each number "a" the number 0 if "a" is irrational, and the number 1 if "a" is rational.

~~Note~~ (i.e. we can write this as: $f(a) = \begin{cases} 0, & \text{if } a \text{ irrational} \\ 1, & \text{if } a \text{ rational} \end{cases}$

Note: A rational number is a number that can be written as a fraction of two integers.
An irrational number is one that cannot be written as a fraction of any two integers.

e.g. $2 = \frac{2}{1}$ (so 2 is a rational #.)

$3/4$ is a rational #.

π is an irrational #.

Ex 5. The following is NOT a function:

$$f(x) = \begin{cases} x^2, & \text{if } 0 \le x \le 2 \\ x^{98}, & \text{if } 1 \le x \le 2 \end{cases}$$

Because if $1 \le x \le 2$, we don't know what to do. f gives 2 possible values: x^2, x^{98} when $1 \le x \le 2$.

So, f is NOT a function because it should only assign a single number for x, not two or more.

· We can also have a composition of ~~2 fund~~ 2 or more different functions. Consider the following examples:

Ex 6 : $f(x) = x^3 + 1$, $g(x) = x^{1/2} + 3$ ← 2 functions.

The composition of these 2 functions:

$f \circ g(x)$ (or also written as $f(g(x))$)

$= f(g(x))$ is a rule that assigns "x" to $(x^{1/2}+3)^3 + 1$.

To see this, notice that:

$$f(g(x)) = (\underbrace{\cdots}_{\text{plug in } g(x) \text{ here}})^3 + 1 = (x^{1/2}+3)^3 + 1.$$

Another possible composition is: $g \circ f(x) = g(f(x))$

$$= (\underbrace{\cdots}_{\text{plug in } f(x) \text{ here}})^{1/2} + 3$$

$$= (x^3+1)^{1/2} + 3.$$

So $\boxed{g(f(x)) = (x^3+1)^{1/2} + 3}$

Ex. 7: $f(x) = \sin(x^2)$: Then $f \circ f(x) = f(f(x))$

$$= \sin((\underbrace{\cdots}_{\text{plug in } f(x) \text{ here}})^2)$$

$$= \sin((\sin(x^2))^2)$$

$$= \boxed{\sin(\sin^2(x^2))}$$

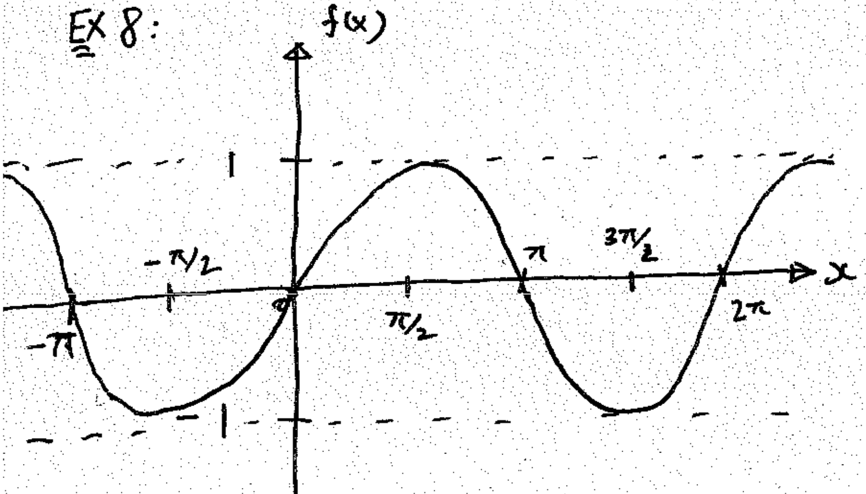
Notice that $f \circ f(x) \neq f^2(x)$

($f^2(x) = \sin^2(x^2)$.)

Graphs: Now that we know what a function is, it helps to **visualize** the function to see the relationship between x and $f(x)$ graphically.

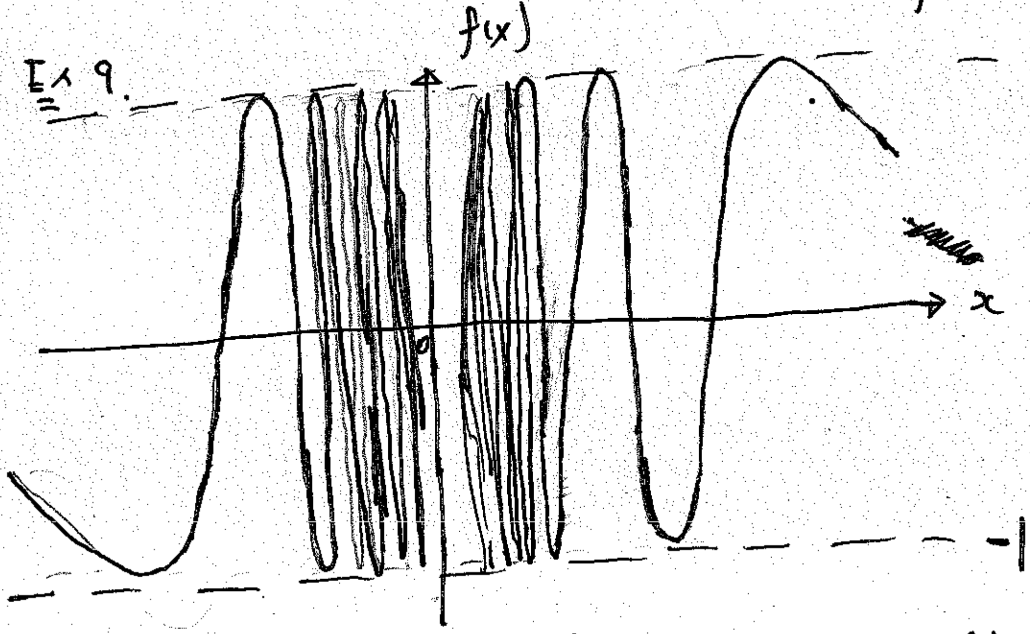
Ex 8:

$f(x) = \sin(x)$.



Ex 9.

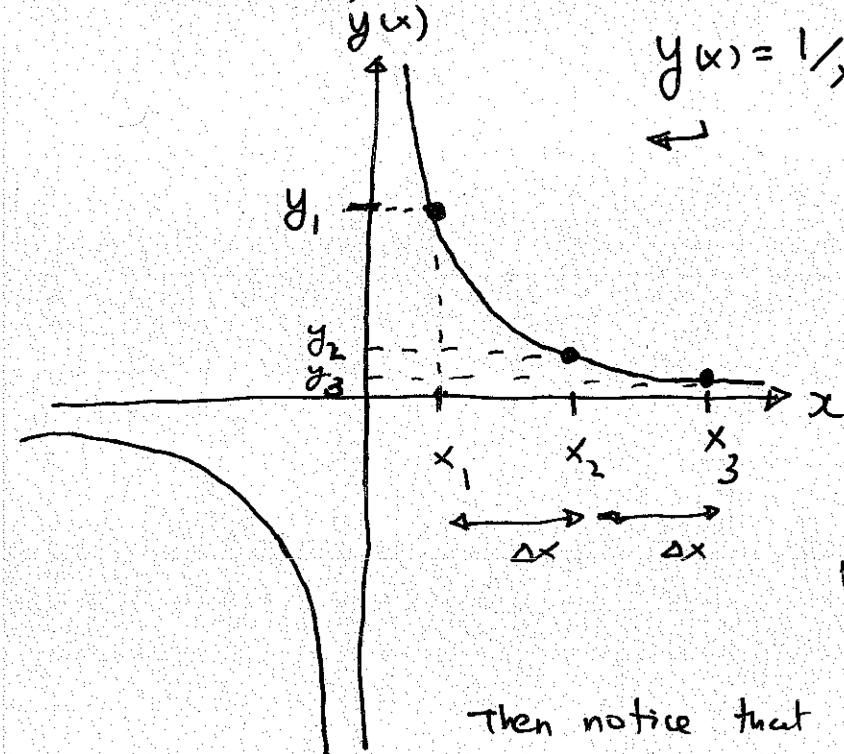
$f(x) = \sin\left(1/x\right)$



- Notice that f is <u>undefined</u> at $x=0$. (because $\left|1/x\right| \to \infty$ as $|x| \to 0$)
- As x approaches zero either from the right $(x>0)$ or left $(x<0)$ of $x=0$, $\sin(1/x)$ wiggles faster & faster. (more "violently".)

To see this, notice that

$y(x) = 1/x$



And $\boxed{\sin\left(\frac{1}{x}\right) = \sin\left(y(x)\right)}$

Consider 3 equally spaced values of x: x_1, x_2, x_3.

[i.e. $x_3 - x_2 = \Delta x$ equally
$x_2 - x_1 = \Delta x$ spaced

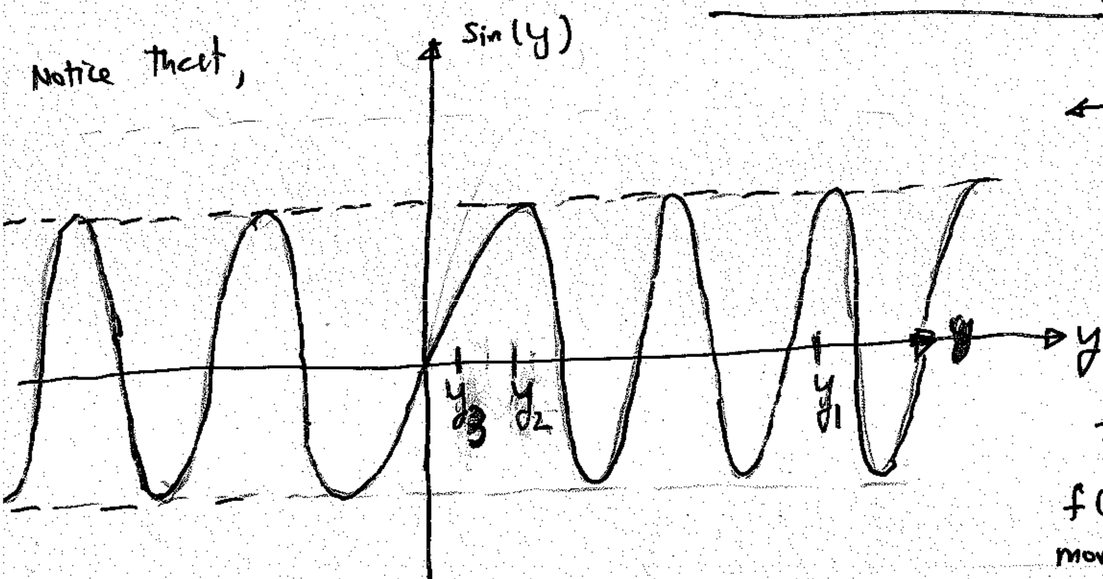
Let $y_1 = y(x_1) = 1/x_1$ $\quad y_2 = y(x_2) = \frac{1}{x_2}$ $\quad y_3 = y(x_3) = \frac{1}{x_3}$

Then notice that unlike the x's, the y's are <u>NOT</u> equally spaced out.

In fact, you can see graphically that $y_1 > y_2 > y_3 > 0$

<u>and</u> $\boxed{y_1 - y_2 > y_2 - y_3}$

Notice that,

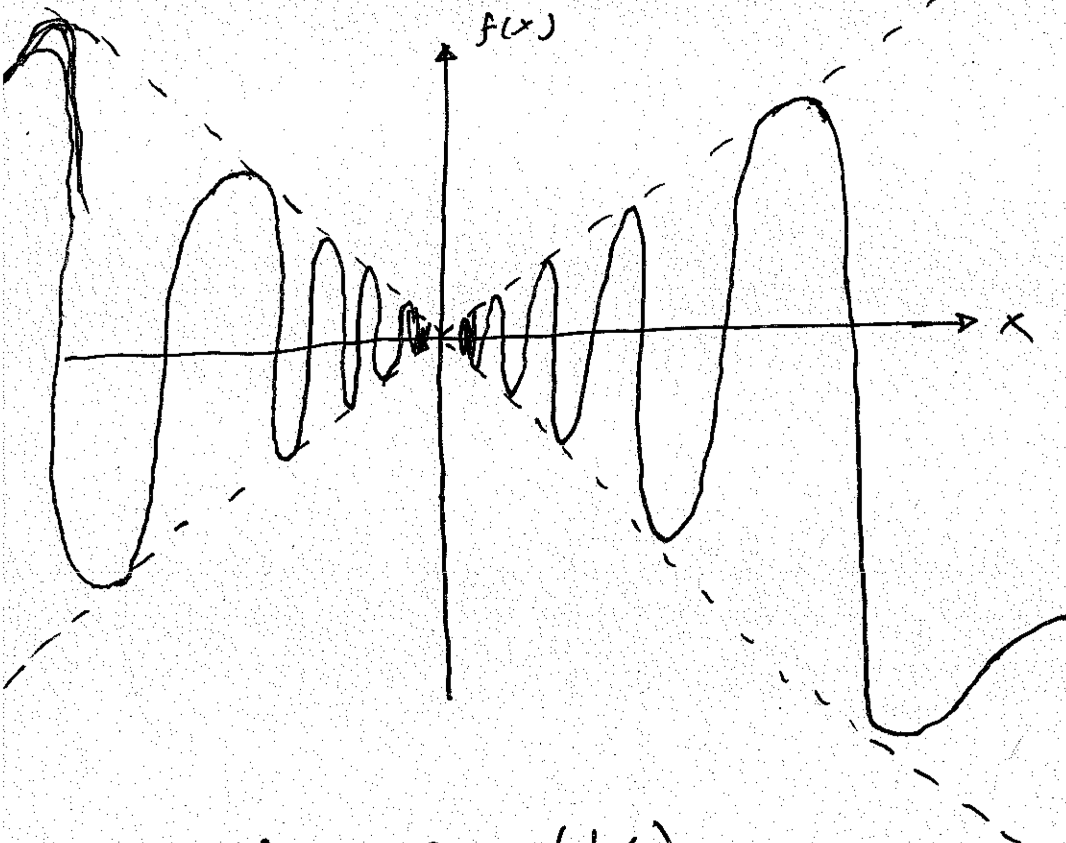


so,
Sin(y) crosses lot more zeros between y_1 & y_2 than it does between y_2 & y_3

And since
$f(x) = \sin\left(\frac{1}{x}\right) = \sin\left(y(x)\right)$

$f(x) = \sin(1/x)$ crosses more zeros between x_1 & x_2 than it does between x_2 & x_3

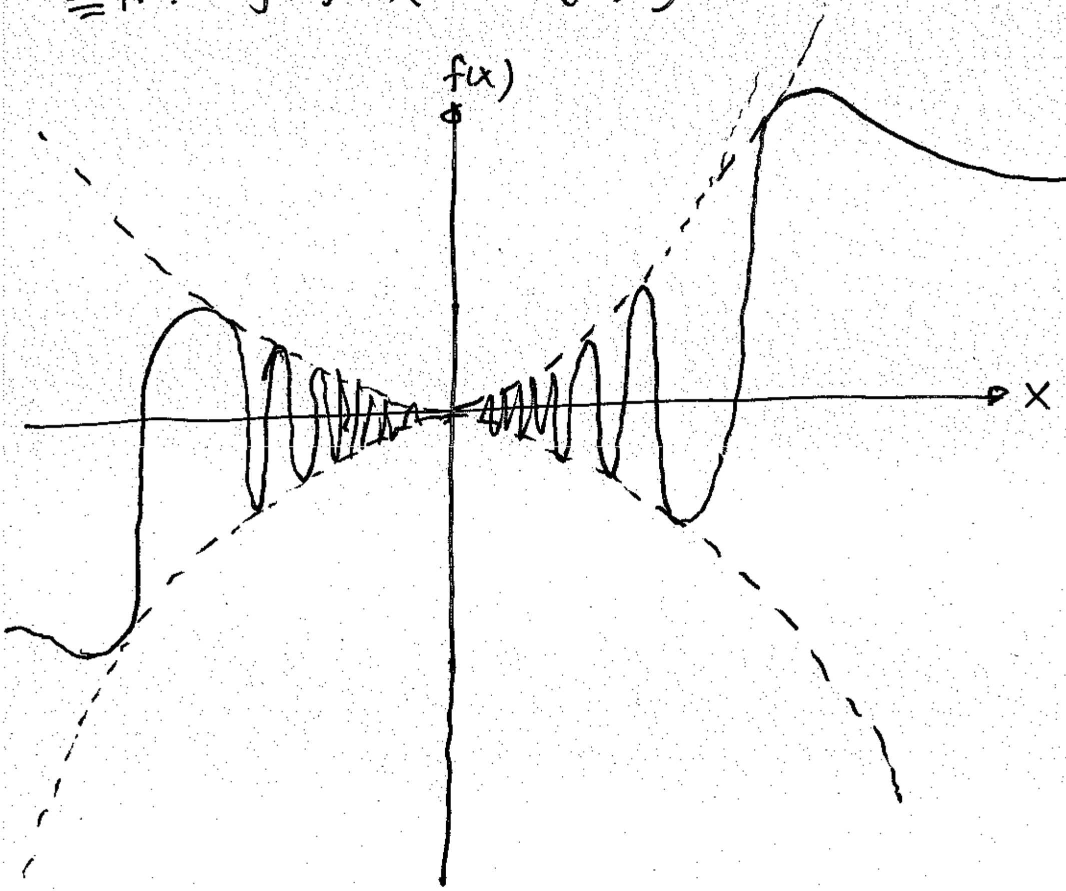
Ex 10 : $f(x) = x \sin\left(\frac{1}{x}\right)$



Dashed lines represent $f(x) = \pm|x|$.

Absolute value sign.

Ex 11 : $f(x) = x^2 \sin\left(\frac{1}{x}\right)$



Dashed curves represent $f(x) = \pm x^2$.

- **Limits**: This tells us how a function $f(x)$ "behaves" near a particular value of x. (but not exactly at x).

Def^n (Physical "loose" definition): The function f approaches the limit "l" near "a", if we can make $f(x)$ as close as we like to "l" by requiring that "x" be sufficiently close to, but unequal to, "a".

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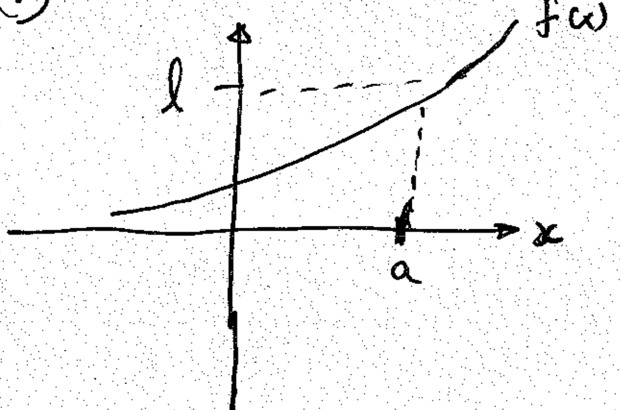

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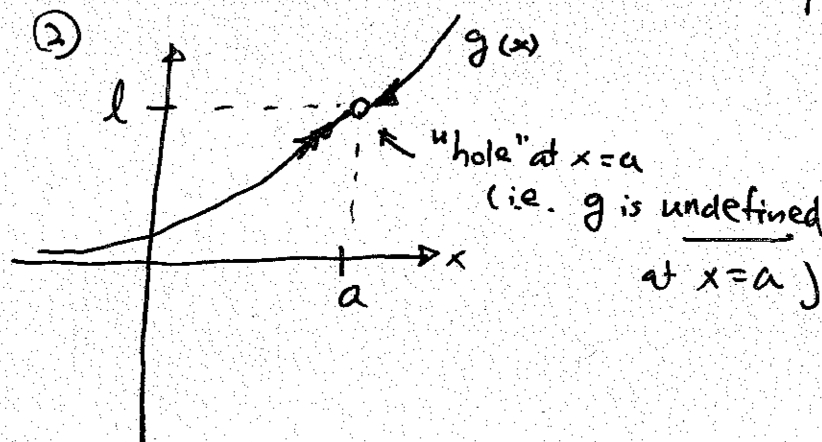

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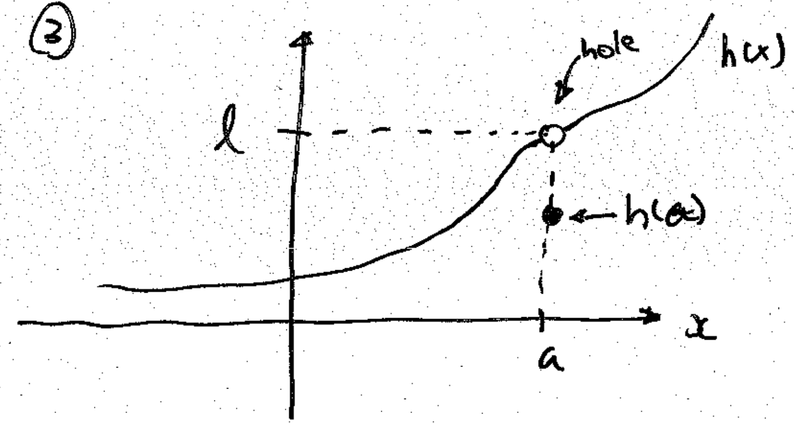

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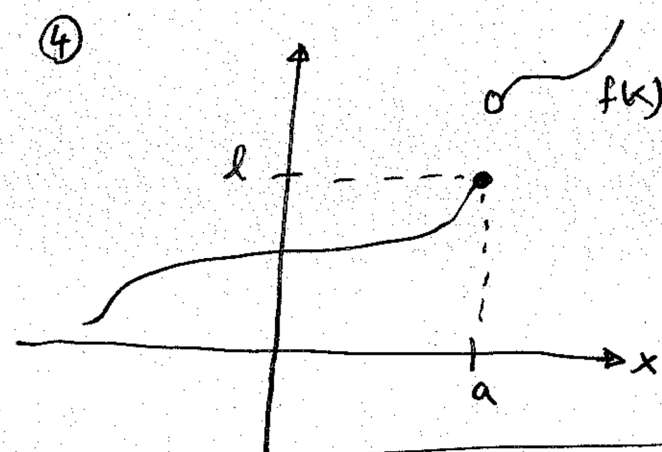

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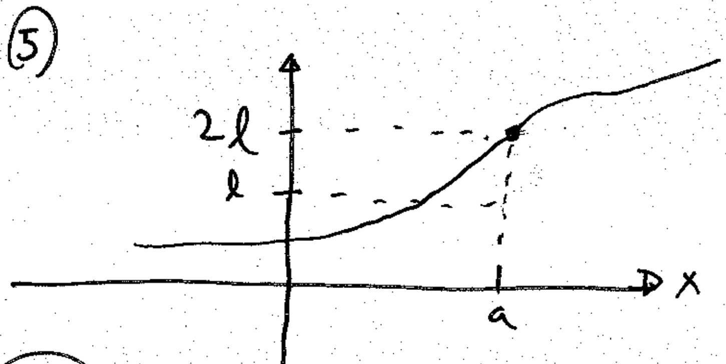


Of these, only (1), (2), and (3) have a function approaching l ~~near~~ at $x=a$.

Looking at (2) & (3), notice that we simply do not care what the function does exactly at $x=a$; only what happens near it.

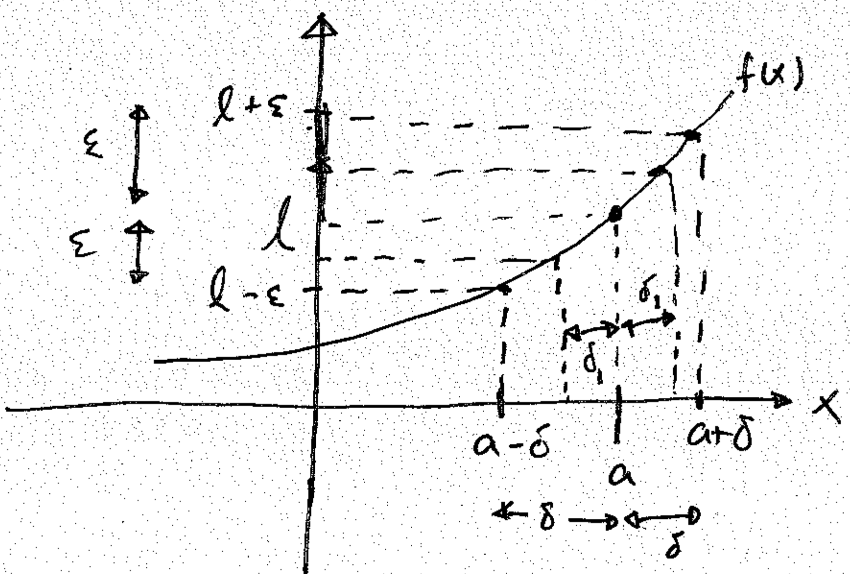
We need to be much more rigorous about our definition of a limit. (In mathematics, we need to be rigorous.)

To motivate a rigorous def^{n} of "limit", look at our physical "loose" def^{n} of limit on previous page.

It says "The function $f(x)$ <u>approaches</u> the limit "l", <u>near</u> "$x=a$"

We can be more specific by asking ① "how closely does $f(x)$ approach l"

and → ② "how ~~near~~ much near "a" does x need to be in order for that to happen?"



← This shows that $f(x)$ is within distance "ε" of l as long as x is within distance δ of a.

(<u>Note</u>: ε and δ are some <u>positive</u> numbers.)

That is:

$$\boxed{|f(x)-l| < \varepsilon \quad \text{as long as} \quad |x-a|<\delta.}$$

But notice also that δ_1 qualifies.

That is: $|f(x)-l|<\varepsilon$ as long as $|x-a|<\delta_1$

In fact you can pick any $\tilde{\delta}>0$ (where $\tilde{\delta}<\delta$) and you'd get $|f(x)-l|<\varepsilon$ for $|x-a|<\tilde{\delta}$.

The smaller you make $\tilde{\delta}$ (some # you pick), the tighter the bound $|f(x)-l|$ becomes.

The idea behind "limit" is that ~~for whatever~~ if indeed $f(x)$ approaches l as x gets near "a", then it means that

for whatever number ε $(\varepsilon>0)$ I give you, you should be able to find some number δ $(\delta>0)$ such that the following condition is satisfied:

$$\boxed{|f(x)-l|<\varepsilon \quad \text{as long as} \quad |x-a|<\delta}$$

*** Formal def^n of limit:**

> The function f approaches the limit l near a means: for every $\varepsilon>0$, there is some $\delta>0$ such that, for all x, if $0<|x-a|<\delta$, then $|f(x)-l|<\varepsilon$.

& called "ε-δ def^n of limit"

Ex 12:

Consider ~~f(x) = x~~.

$$f(x)=\sqrt{2}\,x.$$

Question:

Prove that $\lim_{x\to a} f(x)=\sqrt{2}\,a$

Proof: Looking at the def^n (ε-δ) of limit, ~~we need to find~~ given an arbitrary $\varepsilon>0$, we need to find an appropriately small positive number δ $(\delta>0)$ such that

$$|f(x)-\sqrt{2}\,a|<\varepsilon \quad \text{whenever} \quad |x-a|<\delta.$$

over

so, suppose we're given some positive # $\varepsilon > 0$.

Then, $|\sqrt{2}x - \sqrt{2}a| < \varepsilon$

$\iff \sqrt{2}|x-a| < \varepsilon$

$\iff 0 < |x-a| < \dfrac{\varepsilon}{\sqrt{2}}$. $\quad \left(\dfrac{\varepsilon}{\sqrt{2}} < \varepsilon \text{ since } \sqrt{2} > 1 \right)$

this arrow means "implies" (in both directions)

↑ so we can pick this number to be our δ.

That is, $\boxed{|\sqrt{2}x - \sqrt{2}a| < \varepsilon \text{ whenever } 0 < |x-a| < \delta \quad (\text{where } \delta = \varepsilon/\sqrt{2})}$

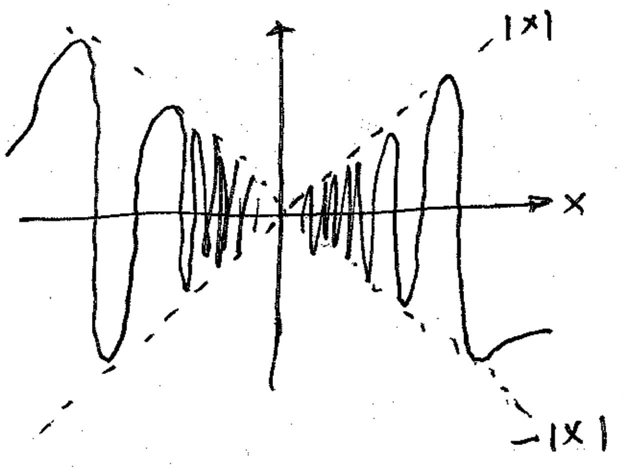
In fact, we can even pick $\delta = \dfrac{\varepsilon}{\sqrt{3}}$ $\left(\text{since } \dfrac{\varepsilon}{\sqrt{3}} < \dfrac{\varepsilon}{\sqrt{2}}\right)$

or $\delta = \dfrac{\varepsilon}{\sqrt{100}}$ $\left(\text{since } \dfrac{\varepsilon}{\sqrt{100}} < \dfrac{\varepsilon}{\sqrt{2}}\right)$

and so on. There are infinite # of possible values of δ we could pick.

But notice that any value larger than $\varepsilon/\sqrt{2}$ as our δ would <u>not</u> work for us; $\dfrac{\varepsilon}{\sqrt{2}}$ is the absolutely largest value of δ we can pick. ∎

<u>EX</u> 13: <u>Question</u>: prove that $\lim\limits_{x \to 0} x\sin\left(\frac{1}{x}\right) = 0$



<u>proof</u>: Given some ε $(\varepsilon > 0)$, we need to find δ $(\delta > 0)$ such that $|x\sin(\frac{1}{x}) - 0| < \varepsilon$ ~~whenever~~ as long as $0 < |x - 0| < \delta$.

→ over

That is, we need δ such that

$$|x \sin(\tfrac{1}{x})| < \varepsilon \quad \text{as long as } 0<|x|<\delta.$$

To do this:

Notice that $0 \le |\sin(\frac{1}{x})| \le 1$ — except when $x=0$. $\sin(\frac{1}{x})$ is <u>undefined</u> at $x=0$; but it's ok. since we only want to know what happens to $\sin(\frac{1}{x})$ when x is <u>near</u> 0, <u>not</u> right at $x=0$.

And so; multiplying both sides by $|x|$ we get:

$$0 \le |x| \cdot |\sin(\tfrac{1}{x})| \le |x| \cdot 1$$

$$\Longleftrightarrow \quad 0 \le |x \sin(\tfrac{1}{x})| \le |x|$$

upper bound on $|x \sin(\frac{1}{x})|$.

so, if $|x| < \varepsilon$, then $|x\sin(\frac{1}{x})| \le |x| < \varepsilon$.

Hence, if we let $\delta = \varepsilon$, then:

$$|x\sin(\tfrac{1}{x})| < \varepsilon \quad \text{as long as } |x| < \delta.$$

($\delta = \varepsilon$ here.)

~~Again,~~ As with <u>eg. 12</u>, this not the only δ we could've picked.

We could have picked $\delta = \varepsilon/2$, for example.

Since: $0<|x-0|<\delta$ means $0<|x|<\varepsilon/2$

$$\Rightarrow 0 < |x||\sin(\tfrac{1}{x})| \le |x| < \varepsilon/2$$

$$\Rightarrow |x\sin(\tfrac{1}{x})| \le |x| < \varepsilon/2 < \varepsilon.$$

means "therefore"

$\therefore$ Indeed, $|x\sin(\frac{1}{x}) - 0| < \varepsilon$ as long as $0<|x-0|<\varepsilon/2$ $(\delta = \varepsilon/2)$.

"ε-δ" proofs like the ones you've just seen on previous pages are not easy. Fortunately, we don't have to go through the "ε-δ" hardly ever to evaluate limits of functions.

To easily evaluate limits of functions, we use the following "limit Theorems" (Thm):

Thm 1: If $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} g(x) = l_2$

then $\lim_{x \to a} f(x) + g(x) = l_1 + l_2$

Reason: Since near "a", $f(x) \underset{\substack{\uparrow \\ \text{"approximately"}}}{\approx} l_1$ and $g(x) \approx l_2$ near

so: $\lim_{x \to a} (f+g)(x) = l_1 + l_2$.

Ex 14: $f(x) = \underbrace{x^3 + x^2 + 3}_{\text{sum of 3 functions.}}$ then $\lim_{x \to 2} f(x) = ?$

Ans:

And, $\lim_{x \to 2} x^3 = 2^3 = 8$

$\lim_{x \to 2} x^2 = 2^2 = 4$

$\lim_{x \to 2} 3 = 3.$

$\underset{\text{"implies"}}{\Longrightarrow}$ $\boxed{\lim_{x \to 2} f(x) = 8 + 4 + 3 = 15}$

Thm 2: If $\lim_{x\to a} f(x) = l_1$ and $\lim_{x\to a} g(x) = l_2$, then

$$\lim_{x\to a} f(x) \cdot g(x) = l_1 \cdot l_2$$

"times"

Reason: Since near "a" $(x \approx a)$, $f(x) \approx l_1$ and $g(x) \approx l_2$

means "approximately near"

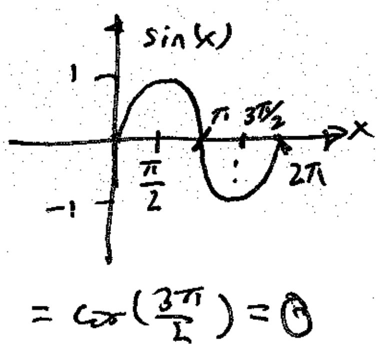
So: $\lim_{x\to a} f(x)\cdot g(x) = l_1 \cdot l_2$

Ex 15: $f(x) = x\sin(x)\cdot\cos(x)\cdot\sec(x)$ ← product of 4 functions.

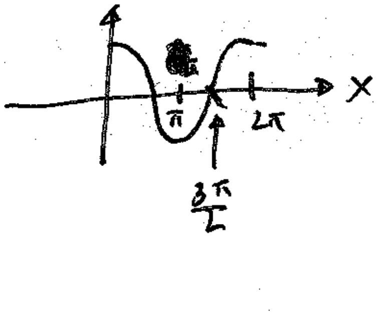
then $\lim_{x\to\frac{3\pi}{2}} f(x) = ?$

Ans: $\lim_{x\to\frac{3\pi}{2}} x = \frac{3\pi}{2}$

$\lim_{x\to\frac{3\pi}{2}} \sin(x) = \sin\left(\frac{3\pi}{2}\right) = -1$



$\lim_{x\to\frac{3\pi}{2}} \cos(x) = \cos\left(\frac{3\pi}{2}\right) = 0$



$\lim_{x\to\frac{3\pi}{2}} \sec(x) = \lim_{x\to\frac{3\pi}{2}} \frac{1}{\cos x} \to \infty.$

blows up since $\cos x \to 0$ as $x \to 3\pi/2$

So....

$\lim_{x\to\frac{3\pi}{2}} f(x) = \left(\frac{3\pi}{2}\right)(-1)(0)(\infty)$

$= \infty$??

No!

In fact, $0\cdot\infty$ so why not $0\cdot\infty = 0$?

How do you know ∞ wins out over zero?

over

The problem here is that we have just blindly plugged in numbers. Whenever you see products involving "0" or "∞", you need to look more carefully. Remember, $\lim_{x\to a} f(x)$ is about how the function behaves when x <u>approaches</u> a but we don't actually care what happens to f(x) when x is <u>exactly</u> equal to a.

Going back to our problem:

$$\lim_{x\to\frac{3\pi}{2}} f(x) = \lim_{x\to\frac{3\pi}{2}} x\sin(x)\cos x \underbrace{\sec x}_{\text{"}1/\cos x\text{"}}$$

$$= \lim_{x\to\frac{3\pi}{2}} x(\sin x)\frac{\cancel{\cos x}}{\cancel{\cos x}}$$

$$= \frac{3\pi}{2}\cdot\sin\left(\frac{3\pi}{2}\right) = \boxed{\frac{-3\pi}{2}}$$ ← Actual answer.

<u>Thm 3</u>: If $\lim_{x\to a} f(x) = l_1$, $\lim_{x\to a} g(x) = l_2$ <u>and</u> $l_2 \neq 0$; then: $\lim_{x\to a}\frac{f(x)}{g(x)} = l_1/l_2$

(For the same physical reason as with the previous 2 theorems.)

But if $l_2 = 0$; then what?

<u>Case 1</u>: If $l_1 \neq 0$ and $l_2 = 0$; then $\lim_{x\to a}\frac{f(x)}{g(x)} = \pm\infty$.

Which sign you use depends on the sign of l_1.

Case 2: If both $l_1 = 0$ and $l_2 = 0$:

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

as long as f & g are both differentiable. [i.e. f' & g' both exist.]

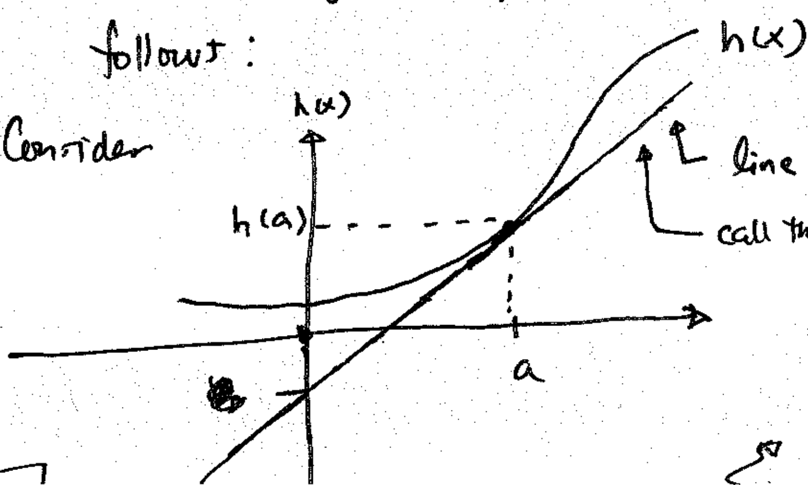
Another way of writing $\frac{df}{dx}$ and $\frac{dg}{dx}$.

Called "l'Hôpital's rule"

Question: Why is l'Hôpital's rule true?

Answer: Not a rigorous proof but the physical reasoning is as follows:

Consider



line tangent to $h(a)$ at $x=a$. (tangent line)

call this $t(x)$.

Equation of tangent line is:

$$\boxed{t(x) = \underbrace{(h'(a))}_{\text{slope of tangent line}}\overbrace{(x-a)}^{\text{at "a"}} + h(a)}$$

This is just the equation of a straight line applied to describe this particular tangent line.

Main Idea: when you're sitting near "a", any function $h(x)$ looks like a straight line passing through $(a, h(a))$ point.

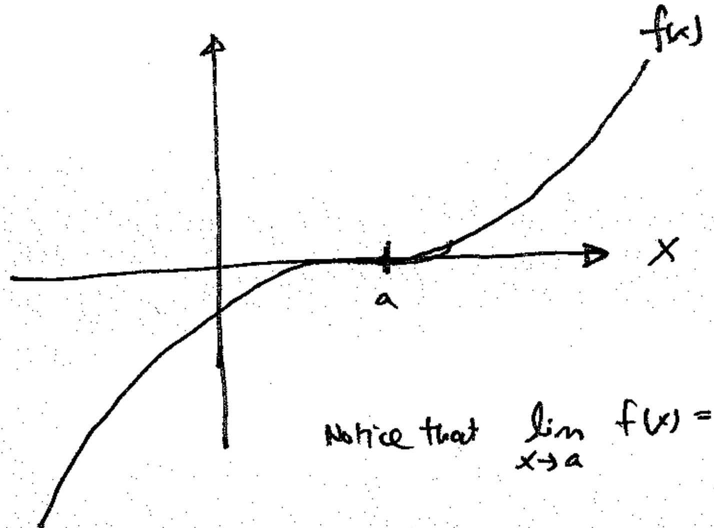
Then near "a": (i.e. when $x \approx a$):

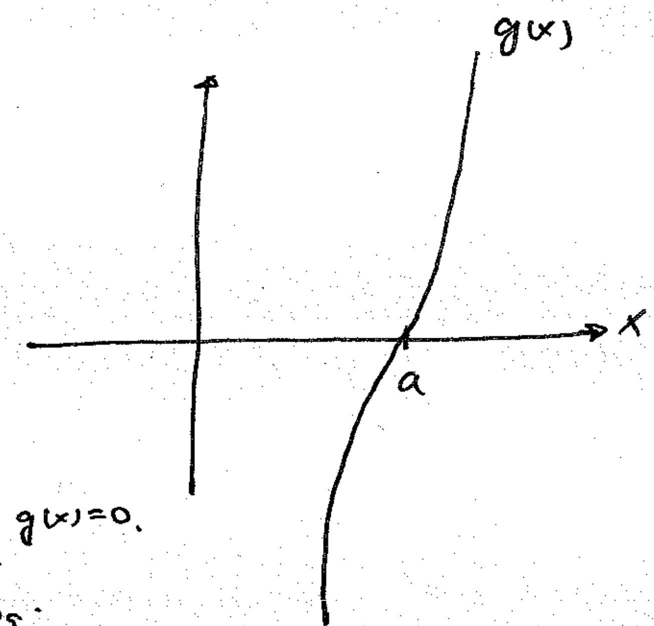
$$h(x) \approx t(x) \approx [h'(a)](x-a) + h(a)$$

$\approx$ means "approximately near"

$$\left(h(x) \approx [h'(a)](x-a) + h(a)\right)$$

So going back to our problem:





Notice that $\lim_{x\to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

Using the approximation from previous pg:

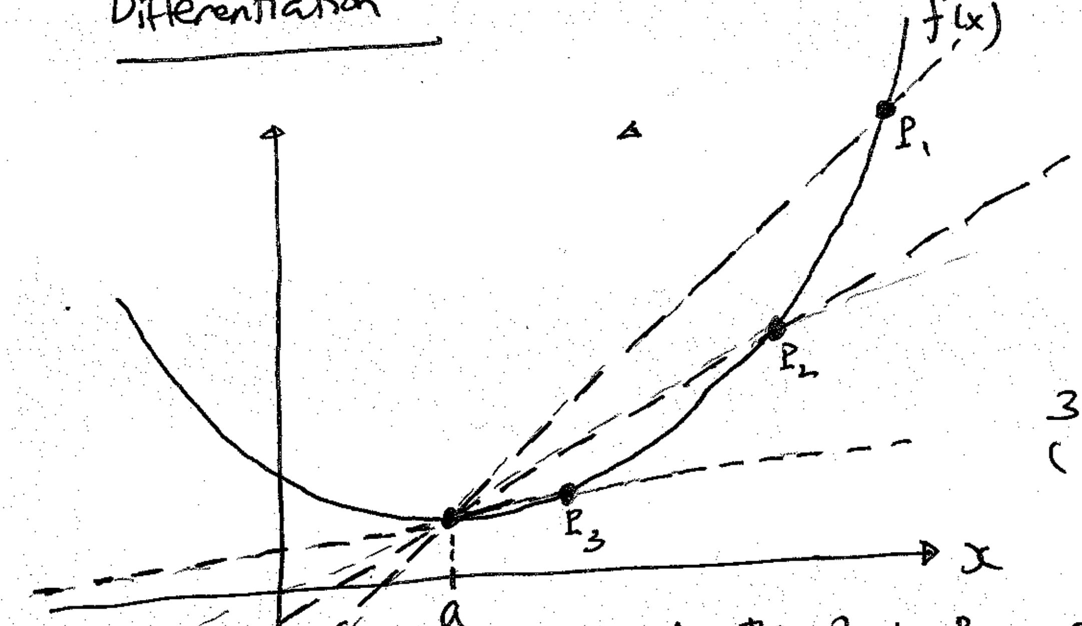
$$\left.\begin{aligned} f(x) &\approx f'(a)(x-a) + \underbrace{f(a)}_{0} = f'(a)(x-a) \\ g(x) &\approx g'(a)(x-a) + \underbrace{g(a)}_{0} = g'(a)(x-a) \end{aligned}\right] \text{ when } x \text{ is near } a.$$

So: ~~Near x~~ when x is near "a": $\dfrac{f(x)}{g(x)} \approx \dfrac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \dfrac{f'(a)}{g'(a)}$

So: $\boxed{\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}}$

This is physically why l' Hôpital's rule holds true.

Differentiation

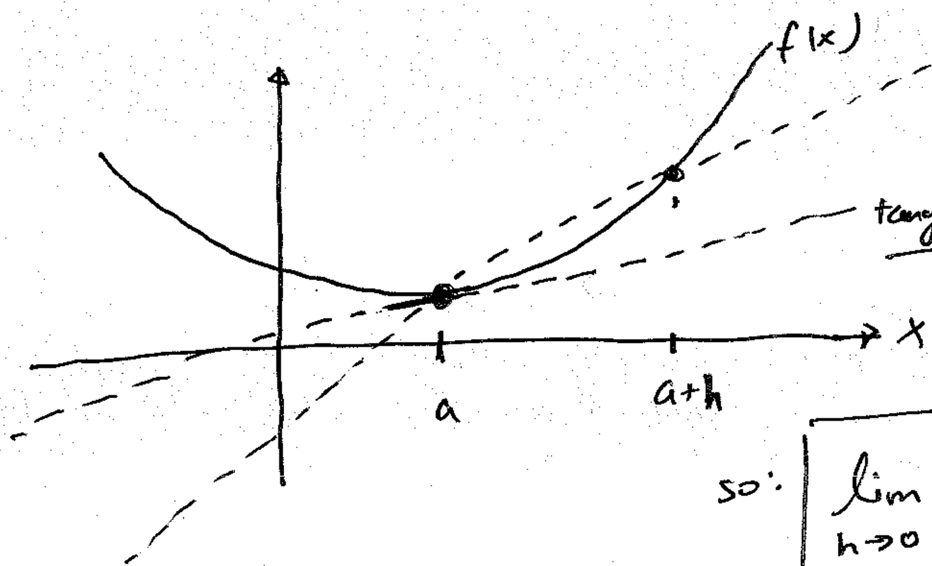


Solid line curve is f(x).

Dashed lines are secant lines.

3 secant lines are drawn (passing through points P_1, P_2, and P_3)

As the point P moves closer and closer to ~~the~~ the point on the curve at $x=a$, the secant line approaches the tangent line at $x=a$.



secant line.
slope $= \dfrac{f(a+h)-f(a)}{h}$

tangent line at $x=a$.

↑ straight line.

"prime"

so: $$\boxed{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)}$$

Also written as $\left.\dfrac{df}{dx}\right|_{x=a}$ sometimes.

Notice that what this is really saying is that any function f(x) that is differentiable at $x=a$ looks like a straight line (tangent line) when you zoom in near $x \approx a$.