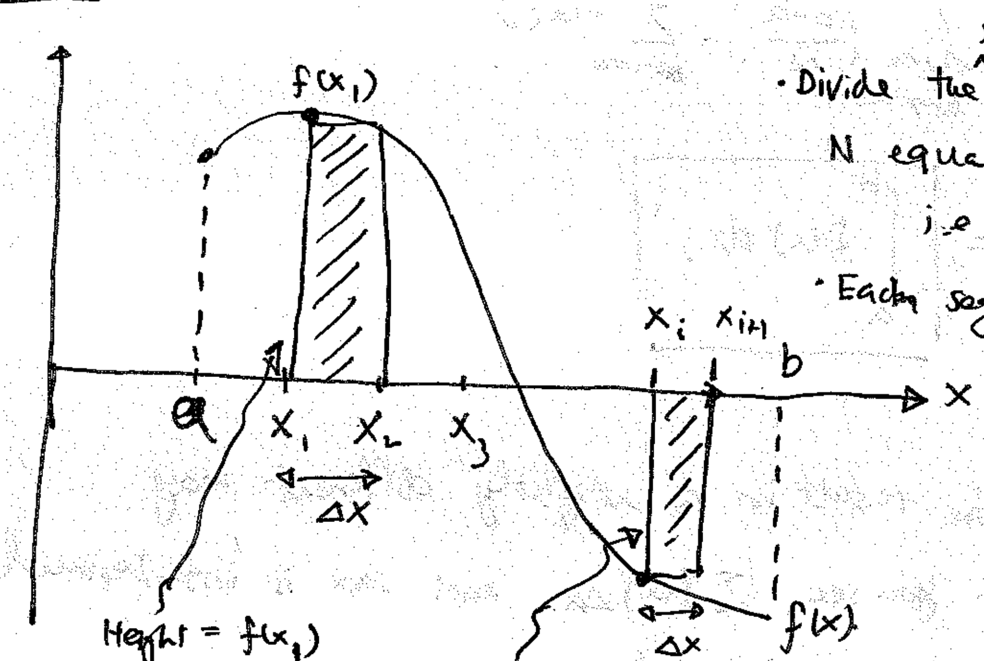
MITES 09 : Calculus II

Lecture # 2

Integrals : Meaning of integral.



- Divide the x interval $[a, b]$ into N equal segments. i.e. $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$
- Each segment has length $\Delta x = \frac{b-a}{N}$

Height $= f(x_1)$
width $= \Delta x$
so "Area" of this Shaded stick is $f(x_1)\Delta x$.

shaded region has "Area" $= f(x_i)\Delta x$
(Note : $f(x_i)\Delta x < 0$ since $f(x_i) < 0$ here.)

Sum up the "area" of each stick (There are N sticks total.) Since we subdivided the x-interval $[a, b]$ into N pieces.)

$$= f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{N-1})\Delta x$$

$$= \sum_{i=0}^{N-1} f(x_i)\Delta x$$

$$= \underbrace{\frac{(b-a)}{N}}_{\text{Approaches zero as } N\to\infty} \underbrace{\sum_{i=0}^{N-1} f(x_i)}_{\text{Approaches } \pm\infty \text{ as } N\to\infty}$$

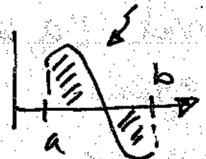
As N gets larger, (i.e. As we make finer and finer partition of the interval $[a,b]$) this becomes more accurate estimation of the "Area" ~~bounded~~ of the region bounded by the curve $f(x)$ and the x-axis.

(I'm writing "Area" with the quotation marks because as shown above for $f(x_i)\Delta x$, "Area" can be negative.)

(These 2 effects counteract one another so that the product of the two approaches a finite number

Sum of "area" of slices $= \frac{b-a}{N} \cdot \sum_{i=0}^{N-1} f(x_i)$

Actual "area" $= \lim_{N \to \infty} \left(\frac{b-a}{N}\right) \cdot \sum_{i=0}^{N-1} f(x_i)$



$\overset{\text{by def}^{\underline{n}}}{=} \boxed{\int_a^b f(x)\,dx.}$

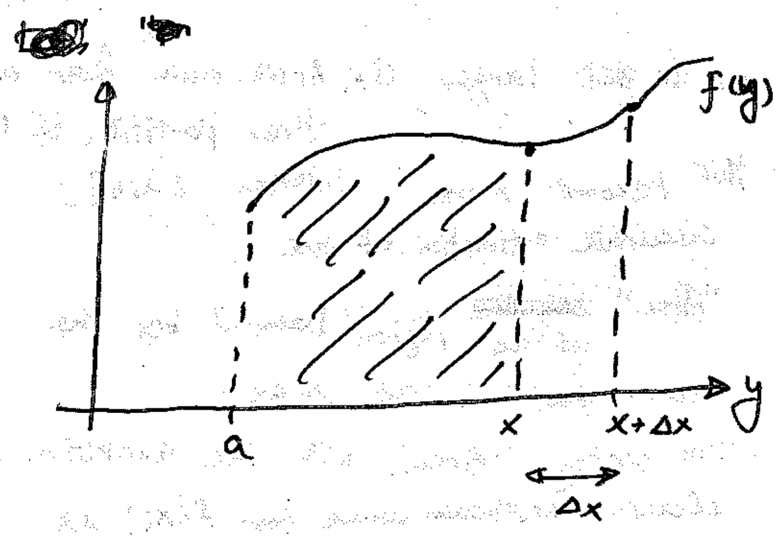
- Let's look at above result in a slightly different way.

~~If you~~ Whenever you see $\sum f(x)\Delta x$. and Δx is infinitesimal quantity, then we write $\Delta x = \underbrace{dx}$. ← means infinitesimally small.

So: we write $\sum f(x)\Delta x \underset{\substack{\text{As } \Delta x \text{ becomes} \\ \text{infinitesimal}}}{=} \sum f(x)\,dx = \int f(x)\,dx.$ (follows from above result.)

← ("stretch" $\sum$ to get $\int$) !!

- Fundamental Theorem of Calculus



Let's define a new function $F(x)$ where:

$$F(x) = \int_a^x f(y)\,dy$$

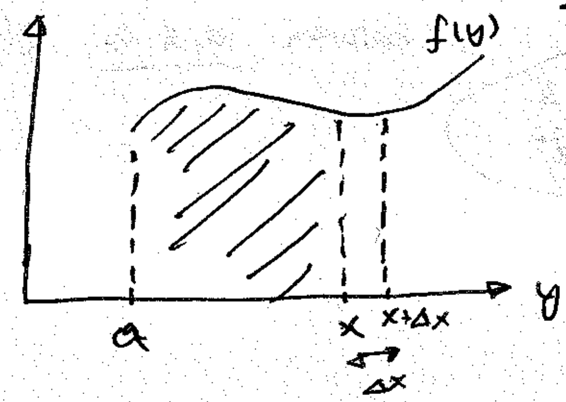
So $F(x+\Delta x) = \int_a^{x+\Delta x} f(y)\,dy$

$$= \underbrace{\int_a^x f(y)\,dy}_{F(x)} + \int_x^{x+\Delta x} f(y)\,dy$$

$$= F(x) + \int_x^{x+\Delta x} f(y)\,dy$$

So: $F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(y)\,dy$

What happens when we make Δx to be infinitesimally small?



← Looking at this picture, notice that the strip of region under the curve between x and $x+\Delta x$ becomes a stick with infinitesimal width (dx).

So:

$$F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(y)\,dy$$

$$= f(x)\,dx = \text{Area of stick between } x \,\&\, x+dx.$$

$\Delta x = dx$. (As Δx becomes infinitesimally small)

So, $F(x+dx) - F(x) = f(x)\,dx$

$\Rightarrow$ (divide by dx) $\underbrace{\dfrac{F(x+dx) - F(x)}{dx}}_{= \frac{dF}{dx} \text{ by def}^{n} \text{ of derivative.}} = f(x)$

So we have: $\boxed{f(x) = \dfrac{dF}{dx}, \quad \text{where} \quad F(x) = \int_a^x f(y)\,dy.}$

↑ This is called the "Fundamental Thm of Calculus"

Notice that the "a" $\int_a^x$ can be any number you desire it to be.

Ex 1: $\int_a^x y^2\,dy = F(x)$

From the theorem above; we need to find F so that $\left.\dfrac{dF}{dx}\right|_y = \underbrace{y^2}_{f(y)}$

Ans: $F(x) = \dfrac{x^3}{3} + C$

↑ Constant.

What is the constant C?

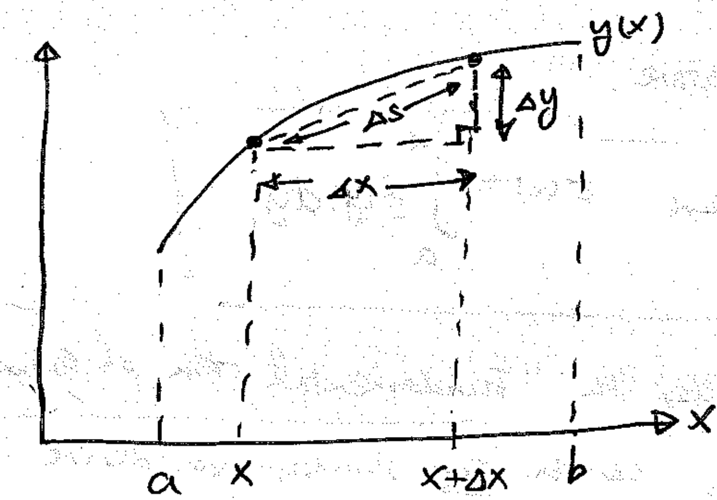
To find it, notice that $F(a) = \int_a^a y^2 dy = 0$

since there is no area enclosed between $\underline{a \; \& \; a}$ (!!)

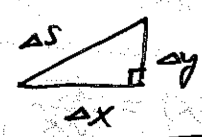
So: $0 = F(a)$
$$= \frac{a^3}{3} + C \Rightarrow C = -\frac{a^3}{3}$$

$$\therefore \boxed{\begin{aligned} F(x) &= \frac{x^3}{3} - \frac{a^3}{3} \\ &= \int_a^x y^2 dy. \end{aligned}}$$

• Let's now calculate length of an arbitrary curve $y(x)$:



← Look at the right angled triangle

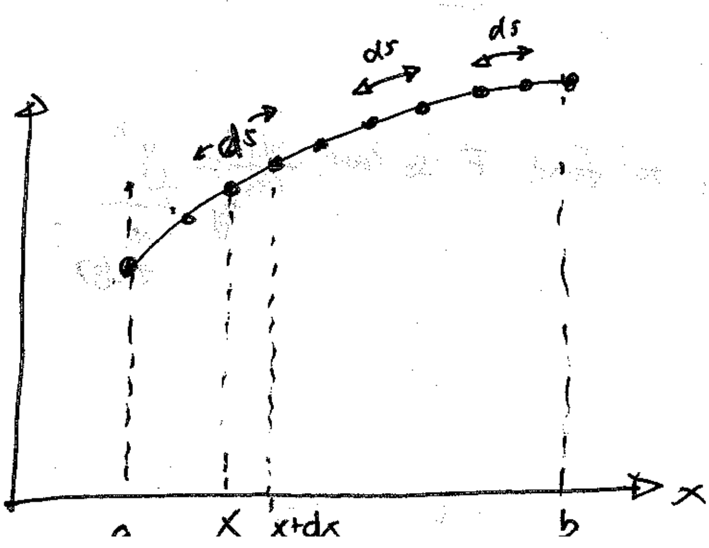


$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

As Δx approaches infinitesimally small width we have

$$ds = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

since $\Delta x = dx$
$\Delta y = dy$.
$\Delta s = ds$.



Sum up the infinitesimal length segment ds starting from a to b to get total length of $y(x)$ between a & b:

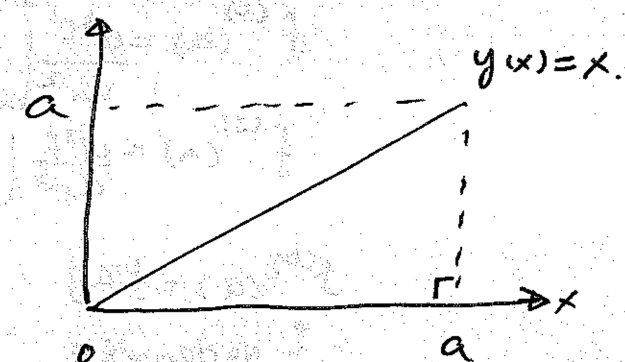
So: Length of $y(x)$ between a & b $= \sum_a^b ds$

$$= \sum_a^b dx \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{\frac{1}{2}}$$

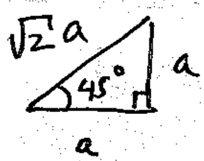
$$= \boxed{\int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

← see (pg 3): There, we said that whenever you see $\sum dx (\dots)$ you write it as $\int dx (\dots)$

Ex 2: Let's check if our formula works for a straight line.



- Length of this line between 0 & a is $\boxed{\sqrt{2}\, a}$.



- check if our formula above gives the same answer:

$\frac{dy}{dx} = 1$

$$\underset{[0,a]}{\text{Length}} = \int_0^a dx \sqrt{1+1}$$

$$= \sqrt{2} \underbrace{\int_0^a dx}_{= x\big|_0^a}$$

$$= \boxed{\sqrt{2}\, a}$$

Indeed, it works! ∎

Taylor series / polynomial.

- Recall that a polynomial of degree n is: $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x_1 + a_0$

  a_i's are coefficients. $(i = 0, 1, 2, \dots, n)$

The idea behind Taylor series/polynomial is that virtually any function $f(x)$ can be approximated by a polynomial if you pick the right coefficients. In particular, the coefficients are derivatives of f evaluated at some arbitrary value $x=a$.

i.e.

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \dots}$$

[Note: $0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$\vdots$ and so on $\vdots$]

[Note: $f'(a) = \left.\frac{df}{dx}\right|_{x=a}$
$f^{(2)}(a) = \left.\frac{d^2f}{dx^2}\right|_{x=a}$
$f^{(3)}(a) = \left.\frac{d^3f}{dx^3}\right|_{x=a}$]

$f^{(0)}(a) = f(a)$
↑ No derivative.

This is the Taylor series of f ~~centered~~ about "a" (or aka. "at a")

We can write it compactly as:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

[Note: "a" can be any constant.
can be zero, 1, π, $\sqrt{\pi}$, etc.]

~~What good is Taylor~~

- What is the Taylor series good for?

Ans: polynomials are easier to work with than other functions. So by representing $f(x)$ in the Taylor series representation, you can do computations easier.

eg. Employing the Taylor series formula from (pg 6), we get:

$$\cos x = \cos(0) + (-\sin(0))(x-0) + \frac{(-\cos(0))(x-0)^2}{2} + \cdots$$

(at $a=0$):

$$= 1 - \frac{x^2}{2} + \cancel{\frac{0 \cdot x^3}{3!}}^{\,0} + \frac{x^4}{4!} + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$ (higher order (power) polynomial terms)

Now, when x is very close to zero: $x^2 \gg x^4$

"much larger than".

To see this, notice that $\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$.

In fact, if $m>n$: $\lim_{x\to 0} \frac{x^m}{x^n} = \lim_{x\to 0} x^{(m-n)} = 0$

since $m>n$

(Note that this would not be true if $n>m$)

So, the moral here is that we can ignore subsequent higher power terms when $x \approx 0$:

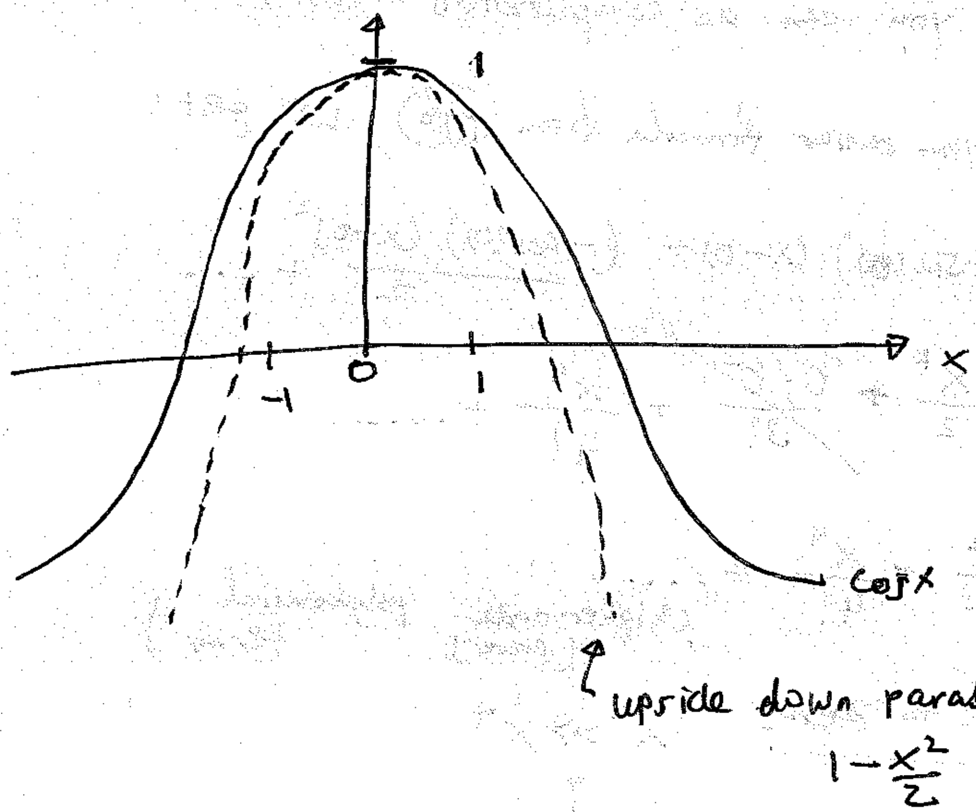
so: $$\cos(x) = 1 - \frac{x^2}{2} + \underbrace{\frac{x^4}{4!} + \cdots}$$

$\approx 1 - \frac{x^2}{2}$ (for $x \approx 0$) Ignore these (since very small compared to x^2)

· Why does it make sense that

$$\cos x \approx 1 - \frac{x^2}{2} \quad \text{when} \quad \underline{|x| \ll 1} \ ?$$

see this graphically:



So, when x is close to zero (our $a=0$), then $1-\frac{x^2}{2}$ is indeed is good approximation of $\cos x$. $\blacksquare$

Some integration techniques

I.) Integration by parts

Notation: $f' = \frac{df}{dx}$ $\quad g' = \frac{dg}{dx}$.

$$\int \frac{d}{dx}(fg)\,dx = \int (f'g + fg')\,dx$$

$$\Rightarrow \quad fg = \int (f'g + fg')\,dx$$
$$= \int f'g\,dx + \int fg'\,dx \quad \Rightarrow \quad \boxed{\int f'g\,dx = fg - \int fg'\,dx}$$

e.g. $\displaystyle\int \underbrace{x}_{g}\,\underbrace{e^x}_{f'}\,dx = xe^x - \int e^x\,dx$
$$= \boxed{xe^x - e^x + C.}$$
$\quad$ constant.

There are 2 very useful tricks in doing integration by parts:

Trick ①: Introduce "1":

Note: In our class "log" means natural log "ln".

e.g. $\displaystyle\int \log x\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\log x}_{g}\,dx = x\log x - \int x \frac{1}{x}\,dx$
$$= \boxed{x\log x - x + C.}$$

Trick ②: use integration by parts to find $\int h$ in terms of $\int h$ again:

e.g. $\displaystyle\int \underbrace{\frac{1}{x}}_{f'}\,\underbrace{\log x}_{g}\,dx = \log x \log x - \int \frac{1}{x}\log x\,dx$

So: $\boxed{\displaystyle\int \frac{1}{x}\log x\,dx = \frac{(\log x)^2}{2}}$.

Technique 2 : Substitution formula

e.g. $\displaystyle\int_a^b \sin^5 x \cos x \, dx$

let $u = \sin x$

$\Rightarrow \dfrac{du}{dx} = \cos x$

$\Rightarrow du = \cos x \, dx$

(and $u_a = \sin a$
$u_b = \sin b$)

$$= \int_{u_a}^{u_b} du \, u^5$$

$$= \frac{u^6}{6} \Big|_{u_a}^{u_b} = \frac{u_b^6 - u_a^6}{6}.$$

$$= \boxed{\frac{\sin^6(b) - \sin^6(a)}{6}}$$

e.g. $\displaystyle\int_a^b \frac{1}{x \log x} \, dx$

let $u = \log x$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$

$\Rightarrow \dfrac{1}{x} dx = du.$

so $\dfrac{1}{x \log x} dx = \dfrac{du}{u}.$

$$= \int_{u_a}^{u_b} \frac{du}{u}$$

$$= \log u_b - \log u_a$$

$$= \log\left(\frac{u_b}{u_a}\right)$$

$$= \boxed{\log\left[\frac{\log b}{\log a}\right]}$$

☒

. 2 very useful trigonometric relationships in doing integrals

Since. $\sin^2 x + \cos^2 x = 1$

. $\cos(2x) = \cos^2 x - \sin^2 x$

we have $\cos(2x) = \cos^2 x - (1-\cos^2 x)$

$= 2\cos^2 x - 1$

And $\cos(2x) = (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$

Thus, $\left\{\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2}. \end{aligned}\right\}$

so, if n is even : $\int \sin^n x \, dx = ?$

e.g. $\int \sin^4 x \, dx = \int \left[\frac{1-\cos 2x}{2}\right]^2 dx$

$$= \int \frac{dx}{4} - \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\underbrace{\int \cos^2 2x \, dx}_{= \int \frac{1+\cos 4x}{2} dx}$$

If n is odd :

~~then~~: $\int \sin^n x \, dx = \int \sin x \left[1 - \cos^2 x\right]^k dx$

(where $2k+1 = n$)

K is integer