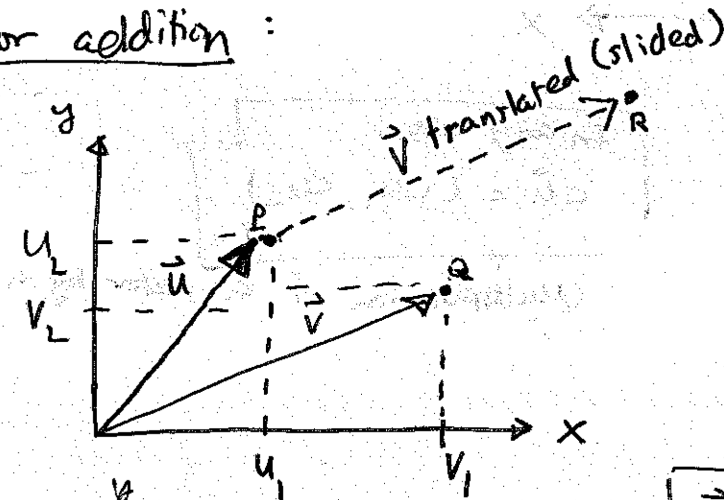
MITES 09 : Calculus II

Lectures 3

Vectors = Arrows indicating · direction and · position.

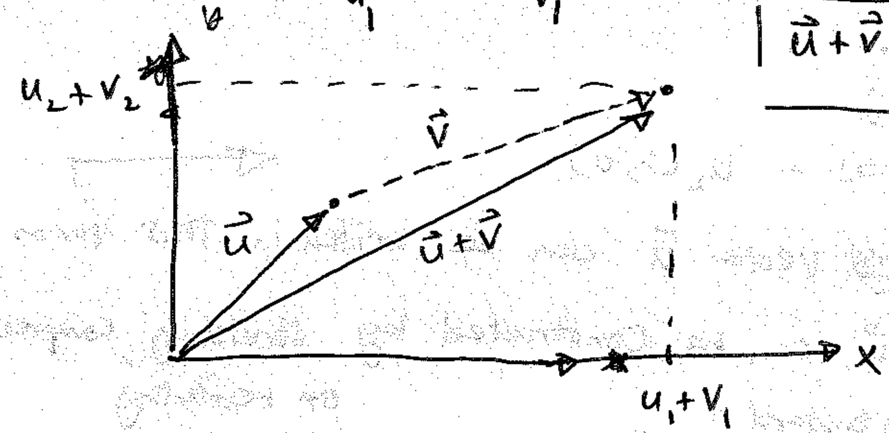
Vector addition :



Point P : $(u_1, u_2) = \vec{u}$ ← vector

Point Q : $(v_1, v_2) = \vec{v}$ ← vector

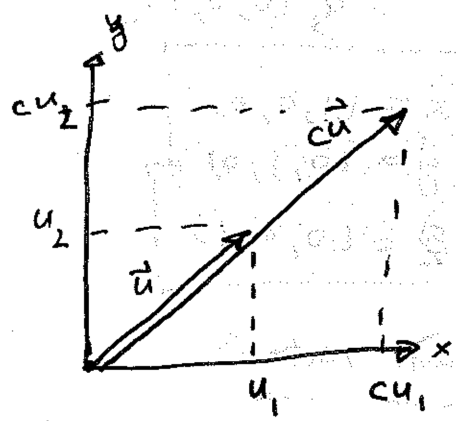
Point R : location of the arrow head of vector $\vec{u}+\vec{v}$.



$$\boxed{\vec{u}+\vec{v} = (u_1+v_1 \,,\, u_2+v_2)}$$

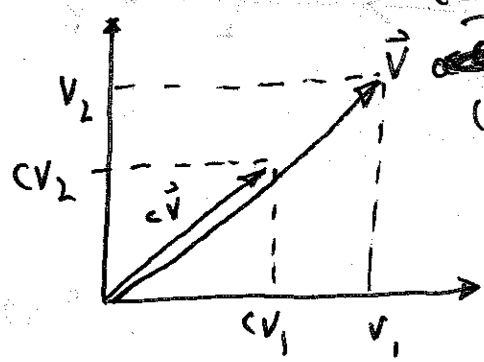
* To add 2 vectors together, move the tail end of one vector to the arrow head of the other vector.
(Notice that it doesn't matter which one you slide since $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.)

· We can also stretch or compress (and reflect) a vector by multiplying it by appropriate number (aka "scalar") c :



(stretch)

$c > 1$

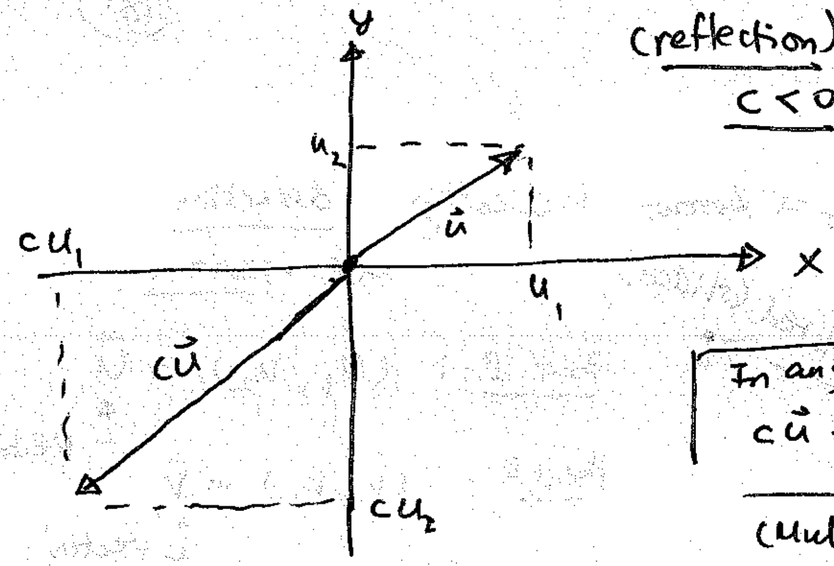


(compress)

$(0 < c < 1)$

(reflection)

$c < 0$.



In any case:
$$\boxed{c\vec{u} = (cu_1, cu_2)}$$

(Multiplication of a vector by a scalar.)

Standard basis vectors

Notice that $\vec{u} = (u_1, u_2)$

$= u_1(1,0) + u_2(1,0)$.

So in 2-dimensions, any vector $\vec{u}$ can be written in this form

(i.e. Any vector $\vec{u}$ can be constructed by stretching, compression or reflecting

the two "standard basis vectors": $(1,0)$, $(0,1)$.)

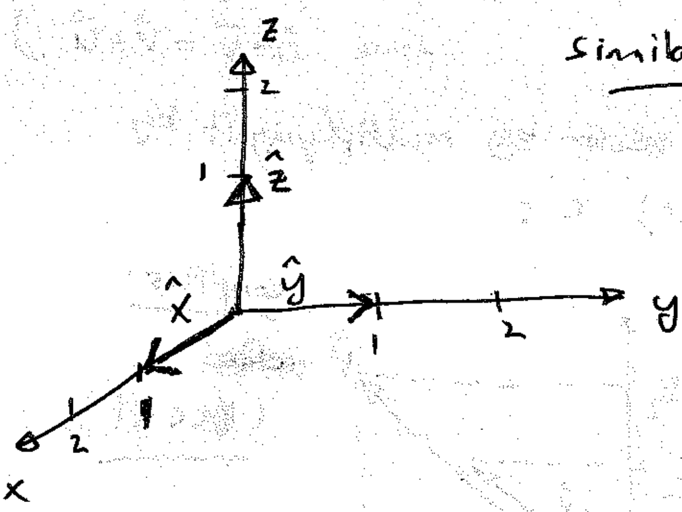
$$\boxed{\begin{aligned}(1,0) &= \hat{x}\\ (0,1) &= \hat{y}\end{aligned}}$$

Similarly, in 3-dimensions:

$\vec{u} = (u_1, u_2, u_3)$

$= u_1(1,0,0) + u_2(0,1,0)$

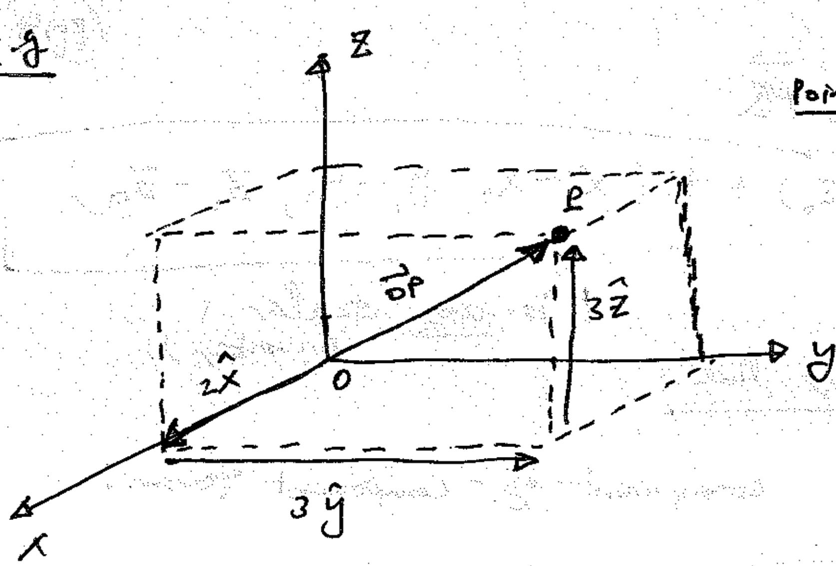
$+ u_3(0,0,1)$



$$\boxed{\begin{aligned}\hat{x} &= (1,0,0)\\ \hat{y} &= (0,1,0)\\ \hat{z} &= (0,0,1)\end{aligned}}$$

"standard basis vectors"

e.g

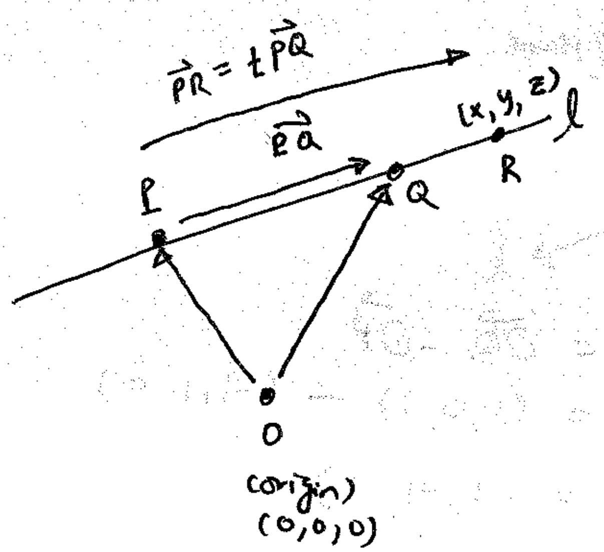


Point P : $(2, 3, 3) = \vec{OP}$ ← vector emanating from origin "O" and with arrow head ending at point P.

$$\boxed{\vec{OP} = 2\hat{x} + 3\hat{y} + 3\hat{z}}$$

(Notice that you can get to point P by following the vector addition rule mentioned on pg1.)

Eq'n of a straight line in any arbitrary direction in 3 dimensional space. (or 2-dim also works too).



Let P & Q be 2 points on the line l.

P : $(x_0, y_0, z_0) = \vec{OP}$

Q : $(x_1, y_1, z_1) = \vec{OQ}$

Then $\vec{OP} + \vec{PQ} = \vec{OQ}$. (see diagram and use our vector addition rule.)

So:

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$
$$= (x_1 - x_0, y_1 - y_0, z_1 - z_0).$$

Consider any arbitrary point R (with coordinates (x, y, z)) that also lies on l.

Then graphically, (see above), notice that $\vec{PR}$ either points in the same direction as $\vec{PQ}$ or in the opposite direction (reflected) of $\vec{PQ}$. In either case: $\vec{PR} = t\vec{PQ}$ (t is some scalar)

Hence,

$$\vec{OR} = \vec{OP} + t\,\vec{PQ}$$

$$\Rightarrow \boxed{(x, y, z) = (x_0, y_0, z_0) + t\,(x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0)}$$

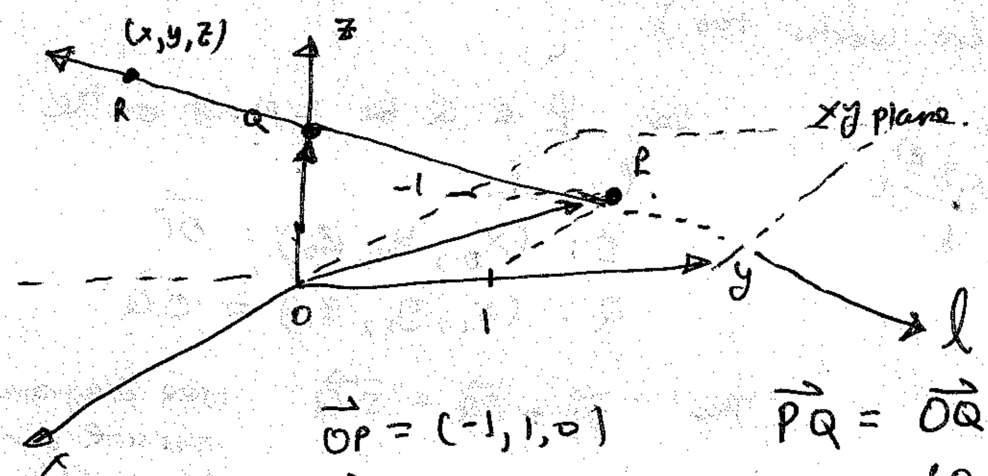
t is <u>any</u> scalar (number).

<u>Eq'n of a straight line.</u>

We can write above in component-by-component form:

$$\boxed{\begin{aligned} x &= x_0 + t\,(x_1 - x_0) \\ y &= y_0 + t\,(y_1 - y_0) \\ z &= z_0 + t\,(z_1 - z_0) \end{aligned}}$$

<u>e.g.</u> Find eq'n of line passing through 2 points: $(-1, 1, 0)$ P
$(0, 0, 1)$. Q



$\vec{OP} = (-1, 1, 0)$

$\vec{OQ} = (0, 0, 1)$

$$\begin{aligned} \vec{PQ} &= \vec{OQ} - \vec{OP} \\ &= (0, 0, 1) - (-1, 1, 0) \\ &= (1, -1, 1) \end{aligned}$$

So: any point $R : (x, y, z)$ on the line l is:

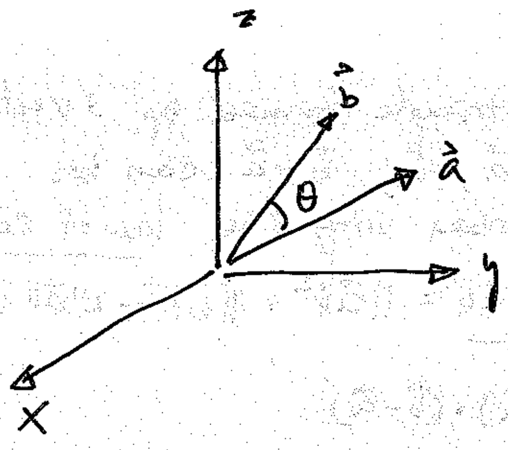
$$\begin{aligned} (x, y, z) &= \vec{OP} + t\,\vec{PQ} \\ &= (-1, 1, 0) + t\,(1, -1, 1) \end{aligned}$$

$\therefore$ $\boxed{\begin{aligned} x &= -1 + t \\ y &= 1 - t \\ z &= t \end{aligned}}$ $\leftarrow$ eq'n of line l. (for <u>any</u> scalar t.)

Inner product : (aka dot product)

1. Helps us find an angle between 2 vectors.

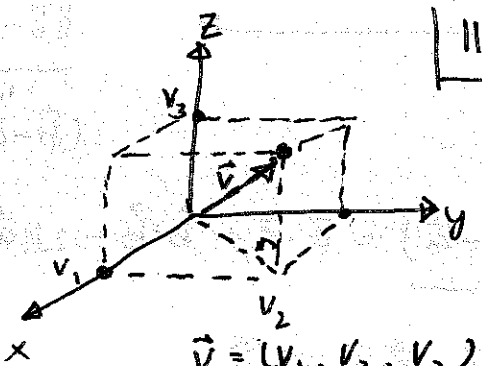
$\theta = ?$



To define dot product, first we define length of a vector :

$$\boxed{\|\vec{V}\| = \sqrt{(V_1^2 + V_2^2) + V_3^2}}$$

straight application of pythagorean theorem.



$\vec{V} = (V_1, V_2, V_3)$

Define dot product between 2 vector $\vec{a}$ & $\vec{b}$ to be:

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3} \quad \leftarrow \text{(dot product)}$$

"dot" product

where $\vec{a} = (a_1, a_2, a_3)$
$\vec{b} = (b_1, b_2, b_3)$.

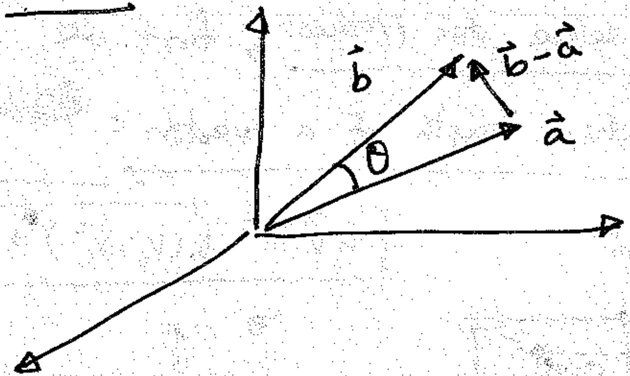
Notice that $\|\vec{V}\|^2 = V_1^2 + V_2^2 + V_3^2$

$$= V_1 V_1 + V_2 V_2 + V_3 V_3$$

$$= \vec{V} \cdot \vec{V}.$$

Claim: $\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta}$ (see diagram on previous page.)

Proof:



← This triangle formed by 3 vectors $\vec{a}$, $\vec{b}$, $\vec{b}-\vec{a}$ can be described using the law of cosines:

$$\underbrace{\|\vec{b}-\vec{a}\|^2}_{=(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})} = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

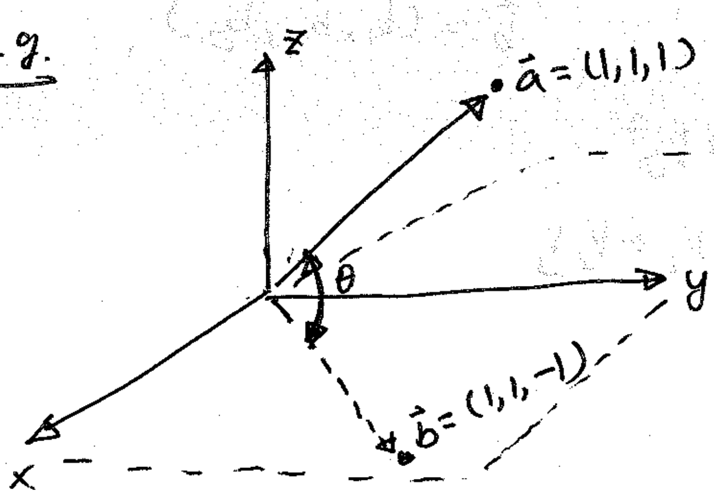
so: $$\underbrace{(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})}_{=\vec{b}\cdot\vec{b}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{a}} = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$\vec{b}\cdot\vec{a}$ and $\vec{a}\cdot\vec{b}$: same

so: $$\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\,\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$\Rightarrow \boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta}$$ proved. Indeed. ☒

e.g.



$\theta = ?$

Ans:

$\vec{a}\cdot\vec{b}$
$= 1+1-1 = 1.$

but also,
$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$
$= \sqrt{3}\sqrt{3}\cos\theta$
$= 3\cos\theta$

Thus: $1 = 3\cos\theta$

$$\Rightarrow \cos\theta = \tfrac{1}{3} \Rightarrow \boxed{\theta = \mathrm{ArcCos}(\tfrac{1}{3})}$$

← In radians

☒