MITES 09 : Calculus II

(Pg 1)

Lecture note # 5

Differentiation in higher dimensions ($\mathbb{R}^n$) (n-dim: $n \geq 2$)

- ~~Scalar-valued function~~ Scalar-valued function : $f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$

x: number (input); x^2: number (scalar) (output)

- ~~[illegible]~~ : $f(x, y, z) = (x^2+y^2+z^2)^{-3/2}$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

(x, y, z): vector (input); $(x^2+y^2+z^2)^{-3/2}$: number (a.k.a. "scalar") (output)

- Vector-valued function : $f(x, y) = (x^2+y^2, xy)$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

(x, y): vector (input); (x^2+y^2, xy): vector (output)

Q: How do we visualize (graph) higher-dimensional functions?

Ans : we can use level sets.

e.g. Suppose $f(x, y, z) = x^2+y^2+z^2$. A level set is a subset of $\mathbb{R}^3$ on which f is constant.

For example, $x^2+y^2+z^2 = 1$ is a level set for f.

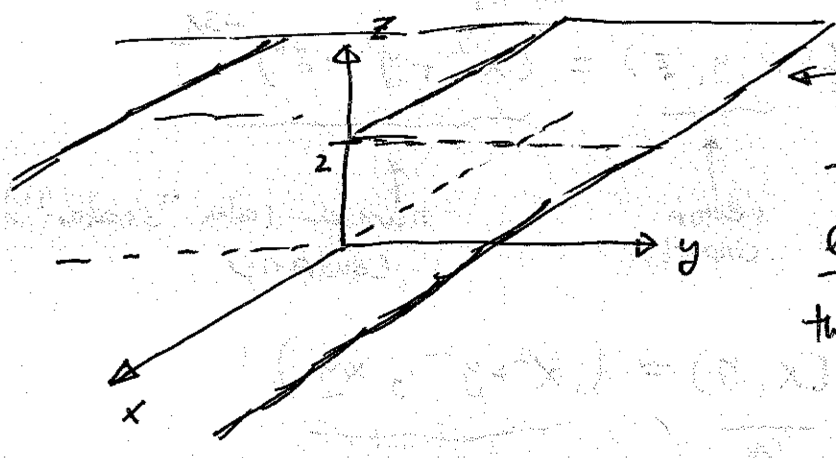
This is just a sphere of radius 1 in $\mathbb{R}^3$.

Formally, a level set is the set of (x, y, z) such that $f(x,y,z) = C$, C is a constant. The behavior or structure of a function is determined in part by the shape of its level sets. So we need to understand level sets of a function to better understand the function itself.

Level sets are also useful for understanding functions of two variables $f(x,y)$, in which case we speak of level curves or level contours.

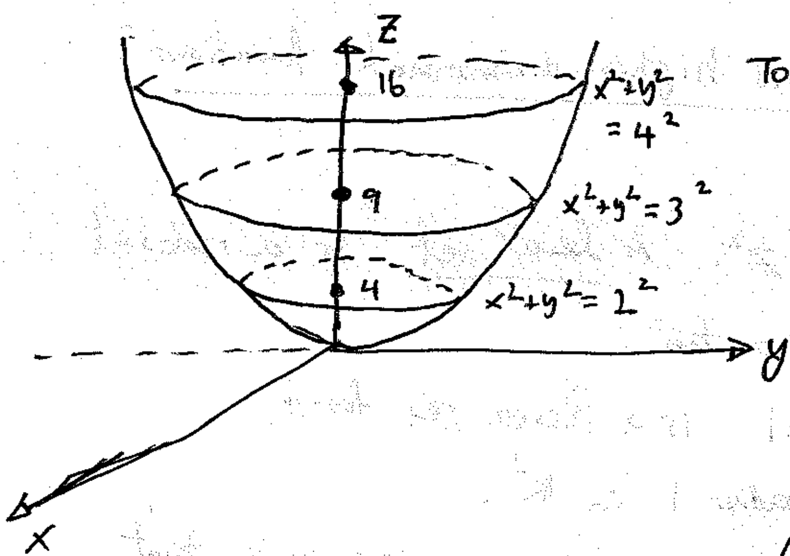
e.g 1: $f(x,y) = 2$. ~~This describes a plane located at z=2~~

This describes the horizontal plane $z=2$ in $\mathbb{R}^3$.



The level curve of value c is empty if $c \neq 2$, and is the whole xy plane if $c=2$.

e.g 2: $f(x,y) = x^2+y^2$ ← Describes paraboloid of revolution.

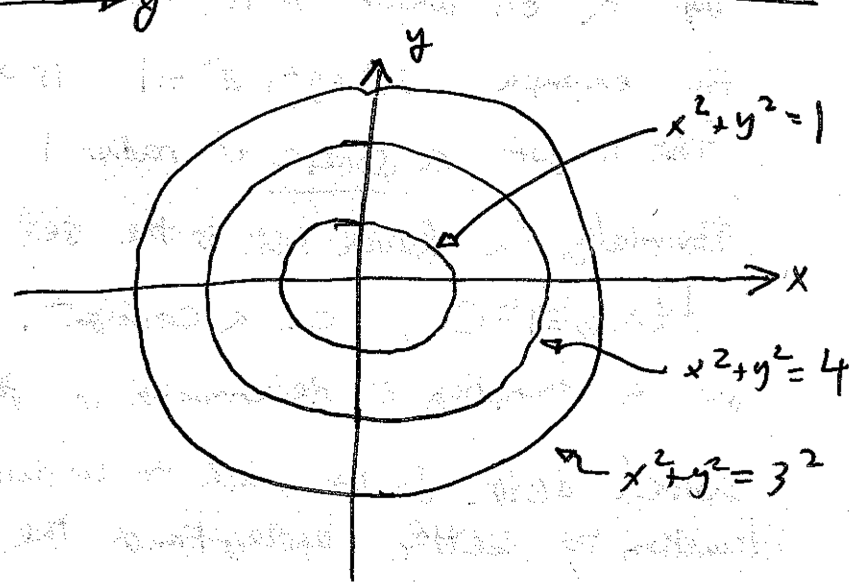


To find level curves of f:

$$\underbrace{x^2+y^2 = c}$$

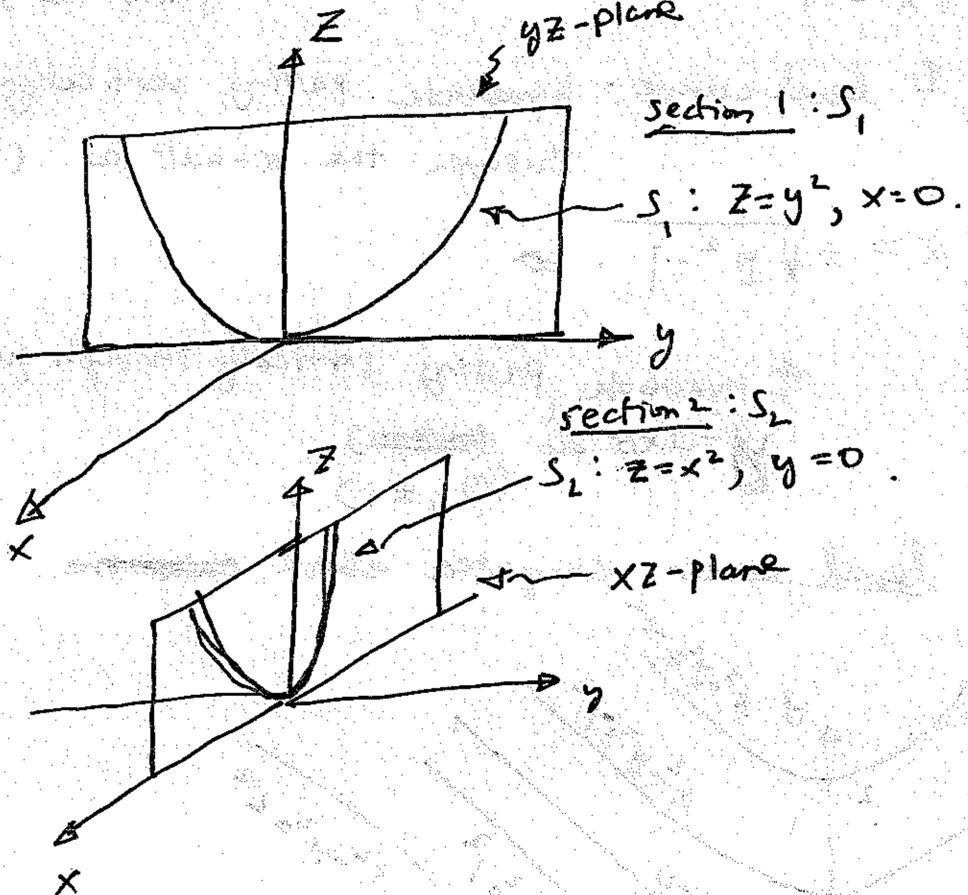
↑ Eq'n of circle with radius c.

sketch of level curves of paraboloid. →



· By a section of the graph f, we mean the intersection of the graph and a (vertical) plane.

e.g. Going back to our paraboloid example: let's take vertical sections of the paraboloid:



e.g. The graph of the quadratic function $f(x, y) = x^2 - y^2$ is called a hyperbolic paraboloid (aka. "saddle"), centered at the origin. How do we sketch this graph?

Ans: To visualize this surface, we first draw the level curves of f.

Consider the values $c = 0, \pm 1, \pm 2^2$.

- For $c = 0$: $x^2 - y^2 = 0 \Rightarrow y = \pm x$

  so this level set consists of 2 straight lines through the origin.

- For $c=1$: $x^2-y^2=1 \Rightarrow y=\pm\sqrt{x^2-1}$

  & This is a hyperbola that passes vertically through the x-axis at the points $(\pm1,0)$.

- For $c=4$:

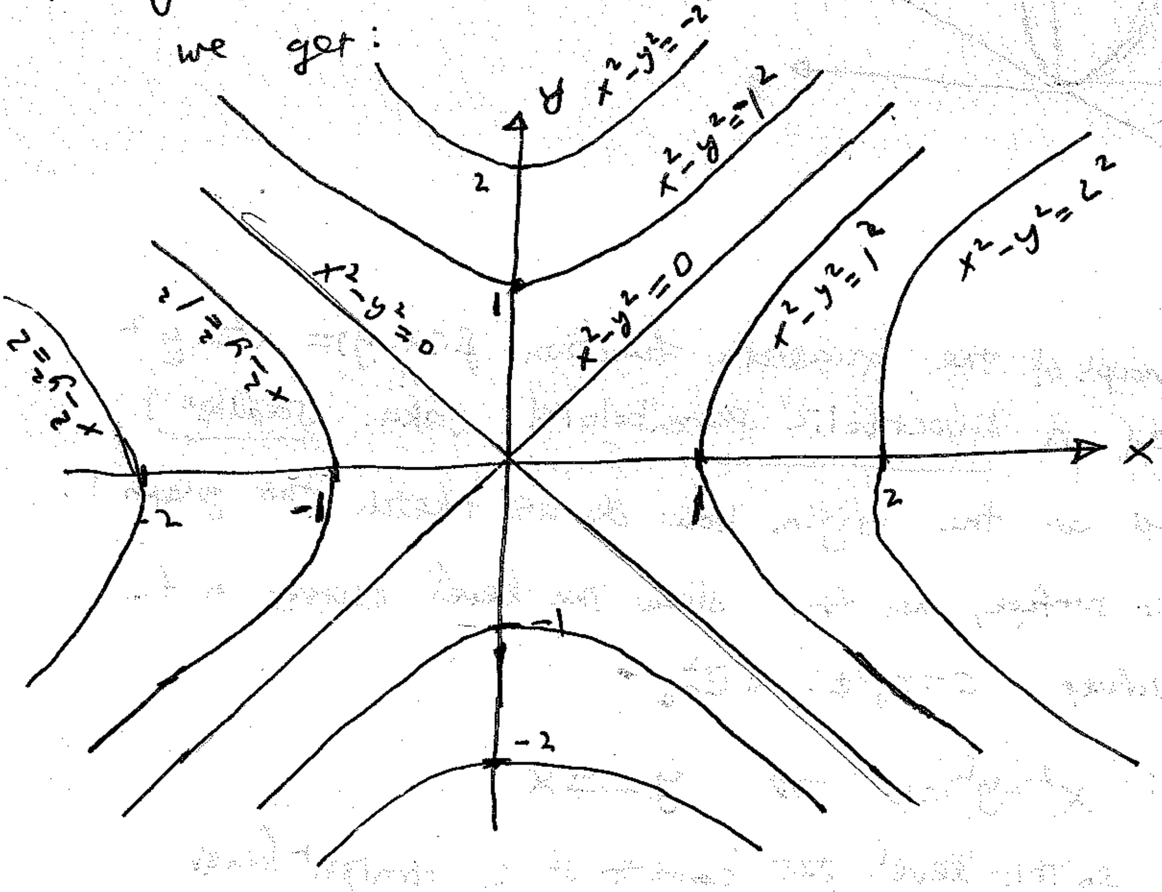
  $y=\pm\sqrt{x^2-4}$

  & level curve: hyperbola passing vertically through the x-axis at $(\pm2,0)$.

- For $c=-1$: $x=\pm\sqrt{y^2-1}$ ⇒

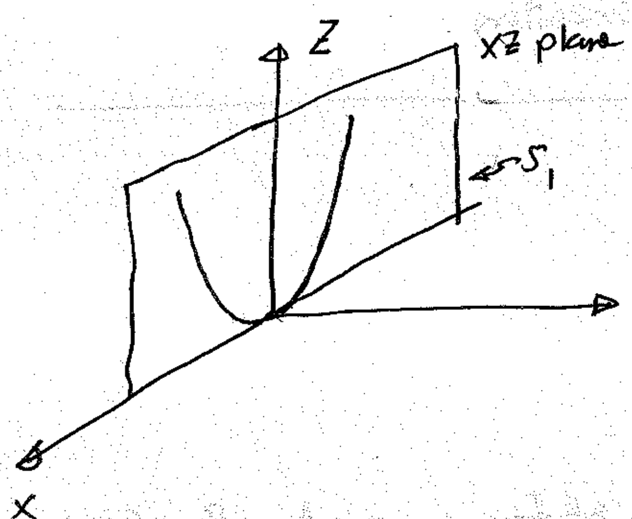
  & Hyperbola passing vertically through the y-axis at ~~(±2,)~~ $(0,\pm1)$.

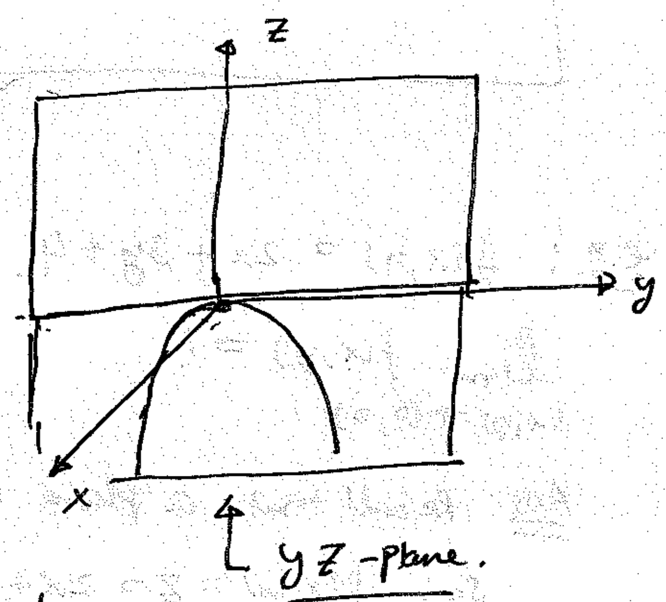
Plotting all the level curves computed above ~~onto one~~ we get:



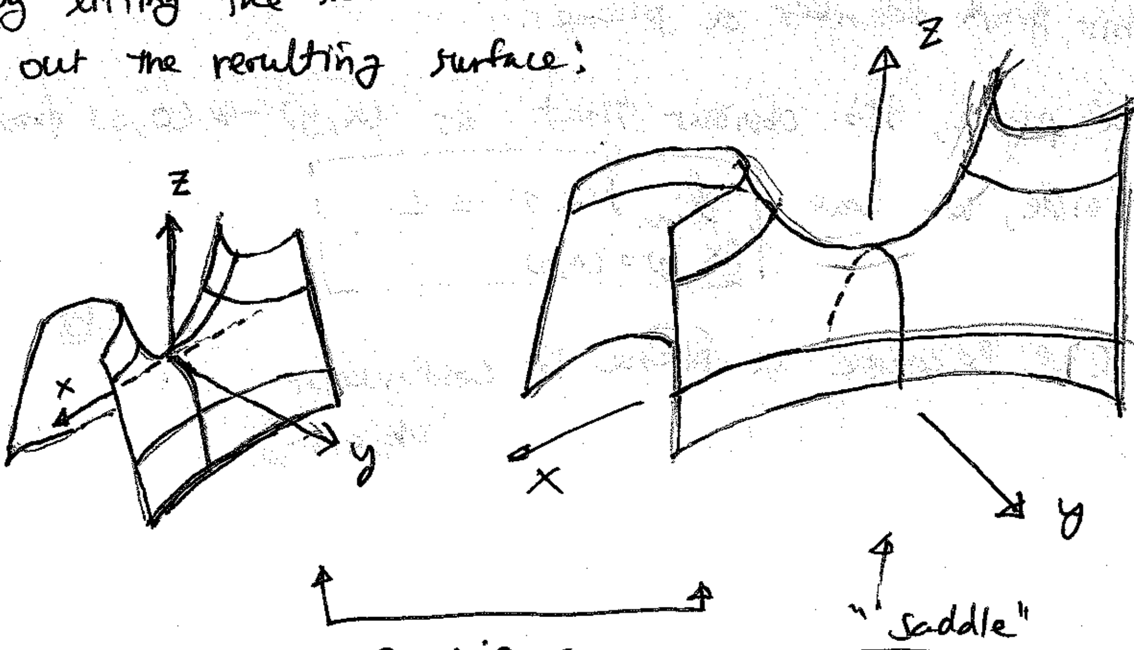
It's not easy to visualize the graph of $f(x,y) = x^2 - y^2$ from these ~~data~~ level curves alone, so we will compute 2 sections to help us: 2 sections are: • ~~xz~~ x plane
• yz plane.

- S_1: xz plane $(y=0)$: $z = x^2$ (parabola)
- S_2: yz plane $(x=0)$: $z = -y^2$ (upside-down parabola)





the graph of f can now be visualized by lifting the level curves to the appropriate heights and smoothing out the resulting surface:



2 views.

"Saddle"

Limits in higher dimensions : For scalar-valued function $f(x,y)$:

$\lim_{(x,y)\to(a,b)} f(x,y) = l$ means that

$f(x,y)$ approaches l as (x,y) approaches (a,b) from any direction.

e.g. : $f(x,y) = 2x + 3y + 4$.

$\lim_{(x,y)\to(0,0)} f(x,y) = ?$

Ans : Recall that a plane has eqn of form: $0 = Ax + By + Cz + D$.

So: $f(x,y) = z = 2x+3y+4$ (i.e., $C=-1$, $D=4$, $A=2$, $B=3$)

so this graph describes a plane.

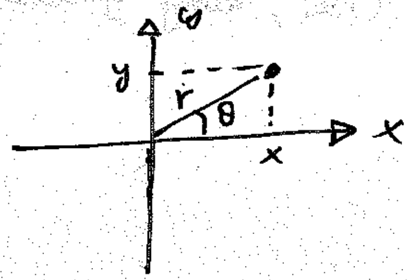
In a plane, it's obvious that as $(x,y) \to (0,0)$ from any side, we have $\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 4}$.

(i.e. Because a plane is continuous object.)

e.g. $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$; $\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

Sol'n : We use polar coordinates to simplify our work.

Notice that $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow x^2+y^2 = r^2\left[\cos^2\theta + \sin^2\theta\right] = r^2.$



$\therefore\ f(x,y) \longrightarrow f(r,\theta) = \dfrac{\sin(r^2)}{r^2}$

Now, notice that as $(x,y) \to 0$, so does $r \to 0$.
and so does $r^2 \to 0$

So: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} f = \lim\limits_{r^2\to 0} \dfrac{\sin(r^2)}{r^2}$

$= \boxed{1.}$ ← since $\lim\limits_{\alpha\to 0} \dfrac{\sin\alpha}{\alpha} = 1$

e.g : Example in which the limit does not exist.

Consider $f(x,y) = \dfrac{x^2}{x^2+y^2}$. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Sol'n : If the limit exists, $\dfrac{x^2}{x^2+y^2}$ should approach a definite value, say "a", as (x,y) gets near $(0,0)$.

In particular, if (x,y) approaches zero along any given path, then $\dfrac{x^2}{x^2+y^2}$ should approach the limiting value a.

over →

- If $(x,y) \to (0,0)$ along the line $y=0$ (i.e. along x-axis),

$$\frac{x^2}{x^2+y^2} \xrightarrow[(y=0)]{} \frac{x^2}{x^2} = 1.$$ So the limiting value is clearly 1.

- But, if (x,y) approaches $(0,0)$ along the line $x=0$ (i.e., along y-axis) the limiting value is

$$\frac{0^2}{0^2+y^2} = 0 \neq 1.$$

Hence: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.

∎

e.g. The function $f(x,y) = \dfrac{2x^2y}{x^2+y^2}$;

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

How? Ans: Notice that

$$\left|\frac{2x^2y}{x^2+y^2}\right| \leq \left|\frac{2x^2y}{x^2}\right| = 2|y|.$$

Thus given $\varepsilon > 0$, choose $\delta = \varepsilon/2$:

Then $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$ implies $|y| < \delta$, and thus

$$\left|\frac{2x^2y}{x^2+y^2} - 0\right| < 2\delta = \varepsilon.$$

∎

Differentiation

- First, define partial derivatives

Consider a scalar-valued function: $f(x_1, x_2, \dots, x_n)$

$[\ f : \mathbb{R}^n \longrightarrow \mathbb{R}\ ]$

Then we define:

$$\boxed{\begin{aligned} \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_j} &= \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_j) - f(\vec{x})}{h}. \end{aligned}}$$

where $\vec{e}_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$

$j-1^{st}$ position

j^{th} position

$j+1^{st}$ position

n^{th}.

Just standard j^{th} basis vector in n-dimensions.

e.g. $f(x,y) = x^2 y + y^3$.

$$\frac{\partial f}{\partial x} = 2xy$$

← to get this, hold y constant (think of it as some number, say 1.) and differentiate only with respect to x.

$$\frac{\partial f}{\partial y} = x^2 + 3y^2.$$

this time, hold x constant and differentiate only with respect to y.

e.g. $Z = \cos(xy) + x\cos y$

sol'n $\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} = -y_0 \sin(x_0 y_0) + \cos y_0$

means evaluated at $(x, y) = (x_0, y_0)$
particular point

$\left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)} = -x_0 \sin(x_0 y_0) - x_0 \sin y_0$.