1) Differential eq'ns:

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{(n-1)}x}{dt^{(n-1)}} + \dots + a_1 \frac{dx}{dt} + a_0 x = 0$$

↑ Ordinary differential eq'n of order n $(n = 1, 2, 3, \dots)$

e.g. $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 5 = 0$ ← differential eq'n of order 2

"Order n" means that the highest derivative you see in the differential eq'n is the n^{th} derivative.

General Theorem: If a differential eq'n is of order n, then its solution $x(t)$ has exactly n constants that depend on the initial condition $(t=0)$.

e.g. $\frac{d^2y}{dt^2} + 3 = 0$.

To solve, we can either guess directly $y(t)$ or use a more systematic method as follows:

Let $x(t) = \frac{dy}{dt}$ $\Rightarrow$ $\frac{dx}{dt} + 3 = 0$ $\left(\because \frac{dx}{dt} = \frac{d^2y}{dt^2}\right)$

$$\Rightarrow dx = -3\,dt$$

$$\Rightarrow \int_{x(0)}^{x(t)} dx = \int_0^t -3\,dt$$

$$\Rightarrow x(t) = x(0) - 3t$$

So: $x(t) = x(0) - 3t$

(Initial ($t=0$) Condition dependent constant)

[There's only one obtained from solving $\frac{dx}{dt} + 3 = 0$ ← 1st order eqn.]

Next, $\frac{dy}{dt} = x(0) - 3t$ (Let's relabel $x(0) = A_1$)

$= A_1 - 3t$

$$\Rightarrow \int_{y(0)}^{y(t)} dy = \int_0^t (A_1 - 3t)\,dt$$

$$\Rightarrow y(t) = y(0) + A_1 t - \frac{3t^2}{2}$$

Initial ($t=0$) condition dependent constant. (call it A_2)

$$\Rightarrow \boxed{y(t) = A_1 t + A_2 - \frac{3t^2}{2}}$$ ← THE sol'n

← 2 constants (A_1, A_2) since 2nd order eqn.

✲ If $y(t)$ is the position of a particle at time t,

then $y(0) = A_2$ ← Initial ($t=0$) position of particle

$\frac{dy}{dt}(t=0) = A_1$ ← Initial ($t=0$) velocity of particle.

∎

E.g. solve $\frac{d^2y}{dt^2} + a^2 y = 0$ ← "simple harmonic oscillator eq'n"

- To solve, let's just guess $y_1(t) = A\sin(at)$.

Then: $\frac{d^2y_1}{dt^2} + a^2 y_1 = \underbrace{-a^2 A\sin(at)}_{\ddot{y}_1(t)} + a^2 y_1$

Check: Does $y_1(t)$ satisfy eq'n?

$= -a^2 y_1(t) + a^2 y_1(t)$

$= 0.$ ← ✓ Indeed, $y_1(t)$ works!

- But $y_1(t) = A\sin(at)$

↑ only one $(t=0)$ constant. But our eq'n has order 2, so the full sol'n to eq'n must have exactly 2 initial condition dependent constants. $(t=0)$

This means we've only found partial sol'n $y_1(t)$.

- Take another guess: $y_2(t) = B\cos(at)$.

Check: $\ddot{y}_2 + a^2 y_2 = -a^2 y_2 + a^2 y_2$

$= 0$

↑ works ✓

[Note: $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$]

So $y_2(t) = B\cos(at)$ is also a sol'n to eq'n.

↑ Another constant.

By linearity: $y_G(t) = y_1(t) + y_2(t)$ is also a sol'n to eq'n.

[you can check that $\ddot{y}_G + a^2 y_G = 0$].

And Notice

$$\boxed{y_G(t) = \underline{A}\sin(at) + \underline{B}\cos(at)}.$$

2 constants.

So we've found ~~all~~ the sol'n to eq'n: $y_G(t)$

"General" sol'n.

Fourier Series: says that pretty much any function $f(t)$ can be represented as a sum of sines and cosines.

Mathematically, it says:

"infinity": Infinite sum.

$$\boxed{f(t) = C_0 + \sum_{n=1}^{\infty} \{ a_n \cos(nt) + b_n \sin(nt) \}}$$

[where C_0, a_n's, and b_n's are constants that have to be correctly picked out.

a_n's $\downarrow$ $\{a_1, a_2, \ldots\}$ b_n's $\downarrow$ $\{b_1, b_2, \ldots\}$

This is a remarkable fact! We can't rigorously prove this but we can, over the next few pages, try to see why this works. To do so, let's first accept that my statement above is true. ~~Then see how to pick ou~~ If so, then how do we assign the right values to C_0, a_n's, and b_n's so that the RHS of above eqn is indeed equal to $f(t)$?

Typical problem: Given $f(t)$, which is a periodic function with period 2π, find its Fourier series.

(i.e. what are the values of a_n, b_n, C_0?)

Algorithm for calculating a_m, b_n, c_0 :

- First, consider

$$\begin{cases} \sin(nt) , & n=1,2,\ldots\ldots,\infty \\ \cos(mt) , & m=1,2,\ldots\ldots,\infty \end{cases}$$

Then $\boxed{\int_{-\pi}^{\pi} \sin(nt)\cos(mt) = 0 \text{ for } \underline{\text{any}} \text{ pair } (n,m).}$

(i.e. $\int_{-\pi}^{\pi} \sin(3t)\cos(4t) = 0$ $\quad (\overset{n}{3}, \overset{m}{4})$

$\int_{-\pi}^{\pi} \sin(3t)\cos(3t) = 0$ $\quad (\overset{n}{3}, \overset{m}{3})$)

Also,

$$\boxed{\begin{aligned} &\int_{-\pi}^{\pi} \sin(nt)\sin(mt) = 0 \quad \text{if } n \neq m \\ \text{and} \quad &\int_{-\pi}^{\pi} \cos(nt)\cos(mt) = 0 \quad \text{if } n \neq m. \end{aligned}}$$

The only case where the integral is not zero is:

$$\boxed{\int_{-\pi}^{\pi} \sin^2(nt)\,dt = \int_{-\pi}^{\pi} \cos^2(nt)\,dt = \pi}$$

Be sure to know why this is equal.

↑ Why?

over

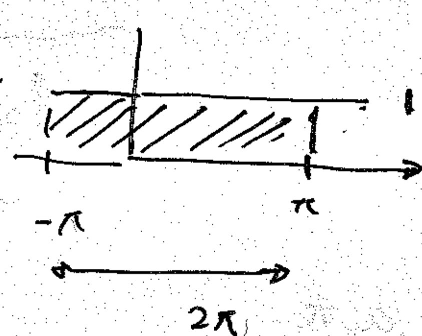
Ans:

$$\int_{-\pi}^{\pi} \sin^2(nt)\,dt = \int_{-\pi}^{\pi} \cos^2(nt)\,dt$$

And $\sin^2(nt) + \cos^2(nt) = 1$

$$\Rightarrow \int_{-\pi}^{\pi} [\sin^2(nt) + \cos^2(nt)]\,dt$$

$$= \int_{-\pi}^{\pi} 1\,dt = 2\pi$$



So: $$2\int_{-\pi}^{\pi} \sin^2(nt)\,dt = 2\pi$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \sin^2(nt)\,dt = \pi}$$

Note: You are asked to prove the boxed relationships on pg6 on Pset #4

Now, these ~~boxed~~ relationships (on pg6) are all we need to calculate the Fourier coefficients $c_0, a_n, b_n \ldots$

Problem: Given $f(t)$, which is a periodic fn with period 2π. find its Fourier series representation

Sol'n

$$f(t) = c_0 + \sum_{n=1}^{\infty} \{a_n \cos(nt) + b_n \sin(nt)\}$$

$$= \ldots + \underbrace{a_k}_{\text{some other coefficient}} \cos(kt) + \ldots + \underbrace{a_n}_{\text{we want to find}} \cos(nt) + \ldots$$

Then:

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \ldots + a_k \overbrace{\int_{-\pi}^{\pi}\cos(kt)\cos(nt)\,dt}^{=0} + \ldots$$

(see pg6)

$$\ldots + a_n \underbrace{\int_{-\pi}^{\pi}\cos^2(nt)\,dt}_{=\pi} + \ldots$$

only this integral (in the ∞ series) survives (i.e., non zero)

$$= a_n \pi$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt} \qquad (n=1,2,3,\ldots\ldots)$$

Similarly, $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$ $(n=1,2,3,\ldots)$

What about c_0?

$f(t) = c_0 + \ldots + a_n\cos(nt) + \ldots$

? This is like $\cos(0\cdot t) = 1$.

So:

$$\int_{-\pi}^{\pi} f(t)\underbrace{\cos(0\cdot t)}_{=1}\,dt = \underbrace{\int_{-\pi}^{\pi} dt\,\cos(0\cdot t)\,c_0}_{=2\pi c_0} + \ldots + \underbrace{\int_{-\pi}^{\pi} a_n\cos(nt)\,dt}_{=0} + \ldots$$

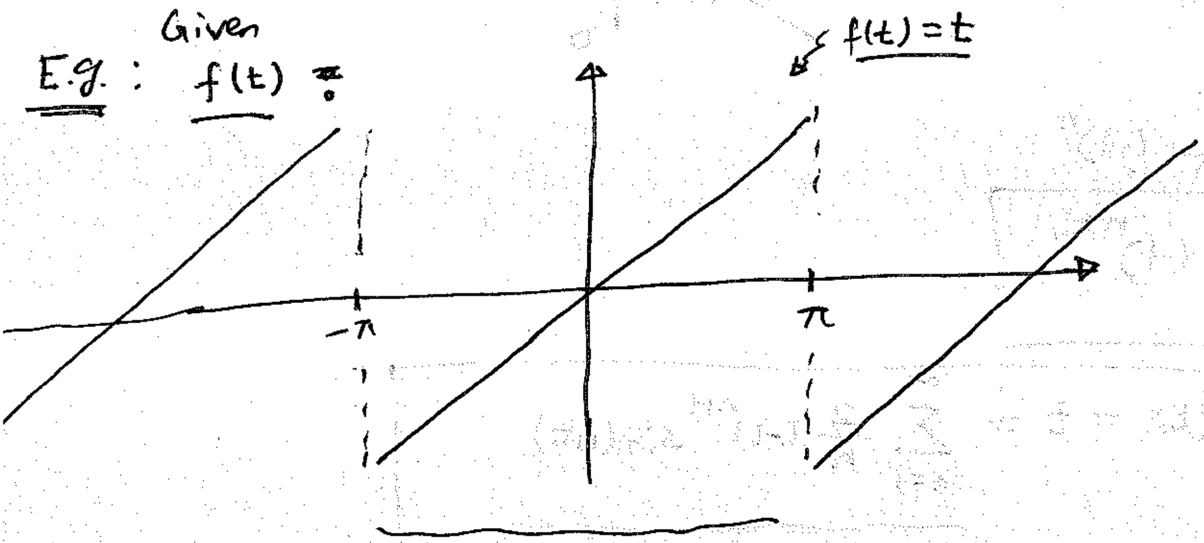
$$\Rightarrow \boxed{c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt}$$

We can use the same formula for c_0 as the one used in computing a_n if we say $c_0 = \frac{a_0}{2}$.

Summary: Fourier series

$$\Rightarrow \boxed{\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nt) + b_n \sin(nt) \} \\ \text{Where, } a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \qquad n = 1, 2, \ldots \end{aligned}}$$

E.g.: Given $f(t)$:



Interested in this part.

(Notice that $f(t)$ is periodic with period 2π.

using above algorithm:
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\cos(nt)}_{\text{odd function on } [-\pi, \pi]}\,dt$$

$= 0.$ for all n.

$\Rightarrow \boxed{a_n = 0}$ (for all n)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\sin(nt)}_{\text{even fn on } [-\pi,\pi]}\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t\sin(nt)\,dt$$

$$= \frac{2}{\pi}\left[\left\{-t\frac{\cos(nt)}{n}\right]\Big|_0^{\pi} - \int_0^{\pi} -\frac{\cos(nt)}{n}\,dt\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n + \cancelto{0}{\frac{\sin(nt)}{n^2}\Big|_0^{\pi}}\right]$$

$$= -\frac{2}{n}(-1)^n$$

$$\Rightarrow \boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

Hence : $\boxed{f(t) = t = \underbrace{\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nt)}_{\text{Fourier series of } f(t).}}$

on $t \in [-\pi, \pi]$.

Notice that

$$b_n = \frac{2}{n}(-1)^{n+1}$$

becomes smaller as n gets larger.

(i.e. $\lim_{n\to\infty} b_n = 0.$)

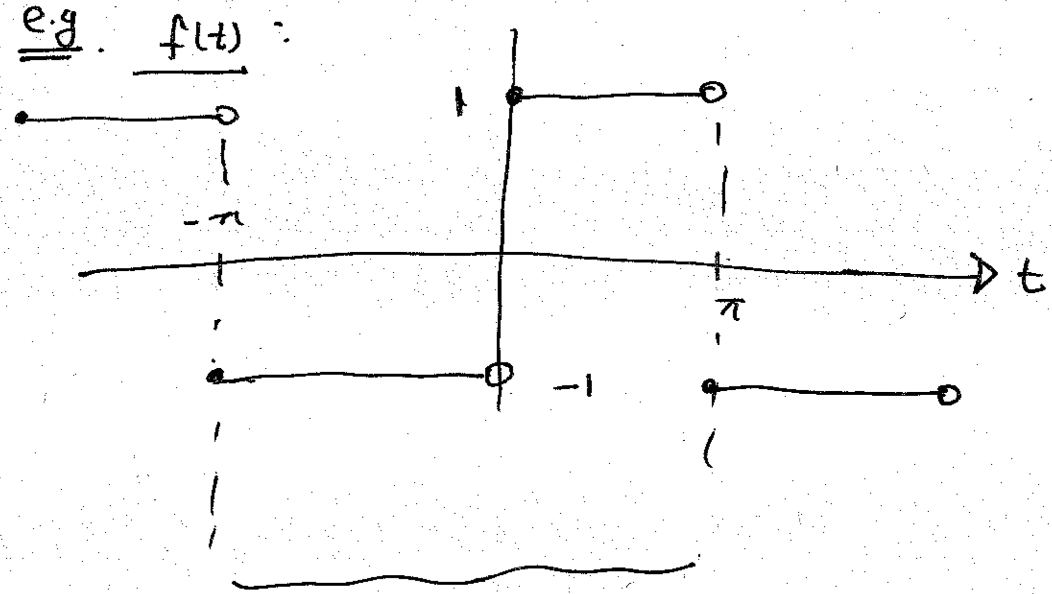
This means that the ~~te~~ terms that appear much later in the infinite series (large n terms), don't matter too much.

Generally, you'll find that this is true:

Terms a_n & b_n with <u>large n</u> (i.e., e.g. a_{1000}, b_{2001}) are <u>very small</u>.

So in practical applications, you can ignore them.

<u>e.g.</u> <u>f(t)</u>:



f(t) in this region is of interest.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)\cos(nt)}_{\text{odd on } [-\pi,\pi]}\, dt = 0$$

; And it turns out (using formula on pg6):

$$b_n = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$= \frac{2}{n\pi}\begin{cases} 2 & ; \text{ if } n \text{ odd} \\ 0 & ; \text{ if } n \text{ even.} \end{cases}$$