1.) (a) $(x,y,z)=(1,-2,-2)+t\,\underbrace{(3,16,-1)}$

$\uparrow$ parallel to this : $\boxed{\vec{v}=(3,16,-1)}$

(b) plane : $x-6y+z-12=0$.

Consider (x_2,y_2,z_2) and (x_1,y_1,z_1) : two points lying on the plane.

$$\Rightarrow \left.\begin{array}{ll} x_2-6y_2+z_2-12=0 & \ldots \text{ Eq'n } ① \\ x_1-6y_1+z_1-12=0 & \ldots \text{ Eq'n } ② \end{array}\right\}$$

so: Eq'n ① – Eq'n ② yields:

$$(x_2-x_1)-6(y_2-y_1)+(z_2-z_1)=0.$$

$$\Rightarrow 0=\underbrace{(1,-6,1)}_{\perp \text{ to plane.}}\cdot\underbrace{(x_2-x_1,\,y_2-y_1,\,z_2-z_1)}_{\text{Any vector lying on the plane}}$$

so: $\boxed{\vec{v}=(1,-6,1)}$

(c) $\vec{v}=(x,y,z)$.

Need: $\vec{v}\cdot\hat{y}=\|\vec{v}\|\underbrace{\|\hat{y}\|}_{=1}\cos\theta=\|\vec{v}\|\cos\theta$

and $\vec{v}\cdot\hat{z}=\|\vec{v}\|\underbrace{\|\hat{z}\|}_{=1}\cos\theta=\|\vec{v}\|\cos\theta$

same since angle θ between $\vec{v}$ and $\hat{y}$ (and $\vec{v}$ and $\hat{z}$) is the same thing.

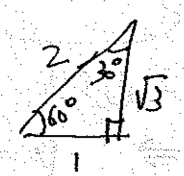
$\Rightarrow \vec{v}\cdot\hat{y}=(x,y,z)\cdot(0,1,0)=y \qquad \vec{v}\cdot\hat{z}=z$

$\Rightarrow \underline{y=z}$: so $(x,1,1)=\vec{v}$

I picked "1" but can be "2", or π or whatever; as long as $y=z$.

Next: $\vec{v}\cdot\hat{x}=\|\vec{v}\|\cos(30^\circ)$

$\Rightarrow x=\|\vec{v}\|\dfrac{\sqrt{3}}{2}$



$\Rightarrow \dfrac{2x}{\sqrt{3}}=\sqrt{x^2+2}$

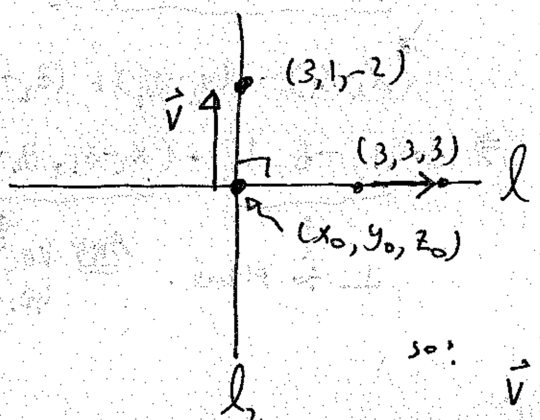
$\Rightarrow \dfrac{4x^2}{3}=2+x^2 \Rightarrow \dfrac{x^2}{3}=2 \Rightarrow x=\sqrt{6}$

$\therefore \boxed{\vec{v}=(\sqrt{6},1,1)}$

2.) (a)
$$\boxed{\begin{aligned}(x,y,z) &= (-1,2,-1) + t\hat{y}\\ &= (-1,2,-1) + t(0,1,0)\end{aligned}}$$

(b)
$$\boxed{\begin{aligned}(x,y,z) &= (0,1,1) + t\{(0,1,0) - (0,1,1)\}\\ &= (0,1,1) + t(0,0,-1)\end{aligned}}$$

(c) $l_2: (x,y,z) = (3,1,-2) + t\vec{v}$: $\vec{v} = ?$



$l: (x,y,z) = (-1,-2,-1) + t(3,3,3)$

$(x_0, y_0, z_0) = (-1,-2,-1) + t_0(3,3,3)$

so:
$$\begin{aligned}\vec{v} &= (3,1,-2) - (x_0, y_0, z_0)\\ &= (3-x_0, 1-y_0, -2-z_0)\end{aligned}$$

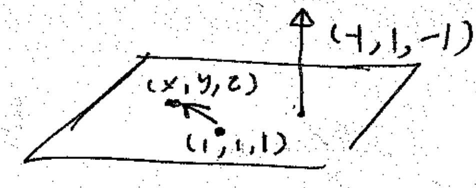
Need:
$$\begin{aligned}0 &= \vec{v}\cdot(3,3,3)\\ &= 3(3-x_0) + 3(1-y_0) + 3(-2-z_0)\\ &= [9+3-6] + [-3x_0 - 3y_0 - 3z_0]\\ &= 6 - 3[-1+3t_0 - 2 + 3t_0 - 1 + 3t_0]\\ &= 6 - 3[-4 + 9t_0]\end{aligned}$$

Solve for t_0: $6 = 9t_0 \Rightarrow \frac{2}{3} = t_0$

$$\begin{aligned}\therefore (x_0, y_0, z_0) &= (-1,-2,-1) + \tfrac{2}{3}(3,3,3)\\ &= (1,0,1)\end{aligned} \Rightarrow \begin{aligned}\vec{v} &= (3-x_0, 1-y_0, -2-z_0)\\ &= (2, 1, -3)\end{aligned}$$

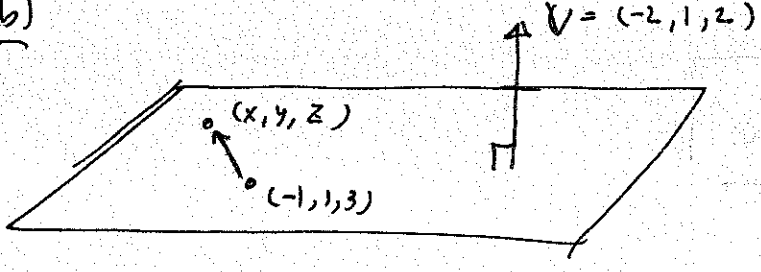
$$\therefore \boxed{l_2: (x,y,z) = (3,1,-2) + t(2,1,-3)}$$

3.) (a) $0 = (-1, 1, -1) \cdot (x-1, y-1, z-1)$



$$\boxed{0 = -(x-1) + (y-1) - (z-1)}$$

(b)



So: $0 = (x+1, y-1, z-3) \cdot (-2, 1, 2)$

$$\Rightarrow \boxed{0 = -2(x+1) + (y-1) + 2(z-3)}$$

(c) Consider the point $(3, 5, -9)$.

Then plugging into eq'n for plane: $A(3) + A(5) - A(-9) - 17A$

$= 3A + 5A + 9A - 17A$

$= 0$

so (3, 5, -9) lies on the plane

Consider another point (x, y, z) that lies on the same plane. Then we have the following 2 eq'ns:

$Ax + Ay - Az - 17A = 0$ ... Eq'n ①

$A3 + A5 - A(-9) - 17A = 0$ ... Eq'n ②

Eq'n ① − Eq'n ②: $\boxed{A(x-3) + A(y-5) - A(z+9) = 0}$

$\hookrightarrow \underbrace{(A, A, -A)} \cdot \underbrace{(x-3, y-5, z+9)} = 0.$

Vector Normal to the plane.

vector lying on the Plane

4.)

(a) $\displaystyle\int \frac{\log x}{x}\,dx$

By substitution: Let $u = \log x$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{dx}{x}$

$= \displaystyle\int u\,du$

$= \dfrac{u^2}{2} + C = \boxed{\dfrac{(\log x)^2}{2} + C}$

constant

(b) $\displaystyle\int (\log x)^3\,dx$ By parts.

$= \displaystyle\int \underbrace{1}_{\text{"}f'\text{"}} \cdot \underbrace{(\log x)^3}_{\text{"}g\text{"}}\,dx = x(\log x)^3 - \int \frac{3\cancel{x}(\log x)^2}{\cancel{x}}\,dx$

$= x(\log x)^3 - 3\left\{x(\log x)^2 - \displaystyle\int \frac{2\cancel{x}}{\cancel{x}}\log x\,dx\right\}$

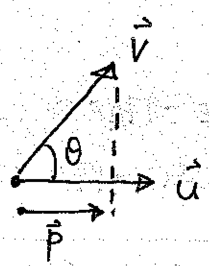
$= x(\log x)^3 - 3\left\{x(\log x)^2 - 2\left[x\log x - \displaystyle\int dx\right]\right\}$

$= \boxed{x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x + C}$

Constant.

5.) (a) $\vec{v} = (1,1,0)$

$\vec{u} = (1,-2,0)$.



Want $\vec{p}$.

$\|\vec{p}\| = \big|\|\vec{v}\|\cos\theta\big|$

Absolute value since $\cos\theta$ can be $\ominus$.

First;

$\vec{v}\cdot\vec{u} = \|\vec{v}\|\,\|\vec{u}\|\cos\theta$

so $\dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|} = \|\vec{v}\|\cos\theta$

$= \dfrac{1-2}{\sqrt{1+4}}$

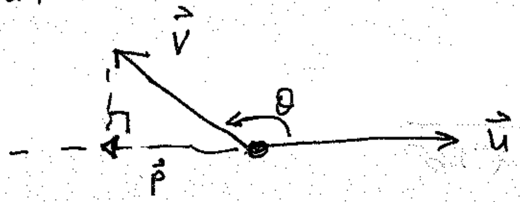
$\Rightarrow \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|} = \dfrac{-1}{\sqrt{5}}$

so $\boxed{\|\vec{p}\| = \dfrac{1}{\sqrt{5}}}$

length of $\vec{p}$.

So, since $\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|} < 0$, the actual picture looks like:

($\theta > 90^\circ$)



$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$ ← unit vector pointing in same direction as $\vec{u}$.

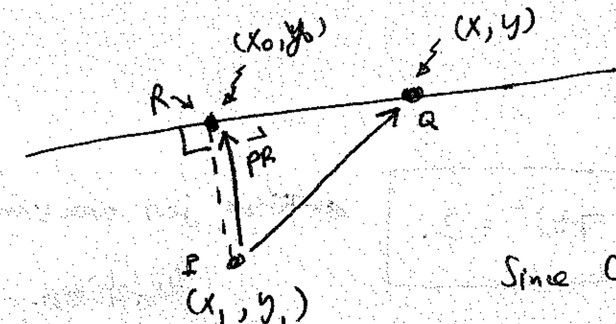
$= \frac{(1,-2,0)}{\sqrt{5}}$

$$\vec{p} = -\frac{1}{\sqrt{5}}\hat{u}$$

$$= -\frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}}(1,-2,0)$$

$$= \boxed{\frac{1}{5}(-1,2,0)}$$ ← orthogonally projected vector.

(b)



$l: ax+by=c$

Want: $\|\vec{PR}\|$ = distance between P and the line.

Since (x_0,y_0) lies on l, we have:

$ax_0+by_0=c$. . . Eq'n ①

Also, since Q: (x,y) lies on l: $ax+by=c$ . . . Eq'n ②

So: Eq'n ② − Eq'n ①: $a(x-x_0)+b(y-y_0)=0$.

$$\Rightarrow \underbrace{(a,b)}_{\text{Vector Normal to } l.}\cdot\underbrace{(x-x_0,\ y-y_0)}_{= \vec{RQ}\ \text{(parallel to line)}} = 0$$

So: (a,b) must be a vector parallel to $\vec{PR}$] $\Rightarrow$ $\vec{PR} = t_0(a,b)$

And: $(x_1,y_1)+t_0(a,b) = (x_0,y_0)$

$\Rightarrow a(x_1+t_0a)+b(y_1+bt_0) = c$ ← since (x_0,y_0) lies on l.

$\Rightarrow ax_1+by_1+(a^2+b^2)t_0 = c$

$$\Rightarrow t_0 = \frac{ax_1+by_1-c}{-(a^2+b^2)}$$

$$\|\vec{PR}\| = \| t_0 (a,b)\|$$

$$= |t_0| \, \|(a,b)\|$$

$$= \frac{|ax_1 + by_1 - c|}{a^2+b^2} \sqrt{a^2+b^2}$$

$$= \boxed{\frac{|ax_1 + by_1 - c|}{\sqrt{a^2+b^2}}}$$ ← distance between point (x_1, y_1) and the line.

6.) $Z(x,y) = -x^2 - y^2 + 3.$

$\vec{c}(t) = (1+4t, 2+7t).$

(a) $Z(x(t), y(t)) = Z(\vec{c}(t))$

$= \boxed{-(1+4t)^2 - (2+7t)^2 + 3.}$ ← Now just one variable (t) function.

(b) $\frac{dZ}{dt} = -2(1+4t)\cdot 4 - 2(2+7t)\cdot 7$

$= -8(1+4t) - 14(2+7t)$

so at $t=3$: $\left.\frac{dZ}{dt}\right|_{t=3} = -8(1+12) - 14(2+21)$

$= -8 \cdot 13 - 14 \cdot (23)$

$= -104 - 322$

$= \boxed{-426}$

$$\begin{array}{r} 14 \\ \times\ 23 \\ \hline 42 \\ 28 \\ \hline 322 \end{array}$$

(c) $\nabla Z(x,y) = \left(\frac{\partial Z(x,y)}{\partial x}, \frac{\partial Z(x,y)}{\partial y} \right)$

$= \boxed{(-2x, -2y)}$

(d) : $\frac{d\vec{c}(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right) = \boxed{(4, 7)}$

(e) $$\frac{dz(\vec{c}(t))}{dt} = \nabla z(x,y) \cdot \frac{d\vec{c}(t)}{dt}$$

$$= (-2x, -2y)\cdot(4,7) = \boxed{-8x - 14y}$$

~~$-8x - 14y$~~

(f) $$T(x,y) = z(0,1) + \frac{\partial z}{\partial x}(0,1)(x-0) + \frac{\partial z}{\partial y}(0,1)(y-1)$$

$$= 2 + [\nabla z(0,1)]\cdot(x, y-1)$$

$$= 2 + (-2(0), -2(1))\cdot(x, y-1)$$

$$= 2 + (0,-2)\cdot(x, y-1)$$

$$\Rightarrow T(x,y) = \boxed{2-2(y-1)} = -2y+4$$

Eq'n of tangent plane

7.) $$u = g(x,y) = xy\, f\left(\frac{x+y}{xy}\right)$$

Then: $$\frac{\partial u}{\partial x} = y\frac{\partial}{\partial x}\left[x f\left(\frac{x+y}{xy}\right)\right]$$

$$= y\left[f\left(\frac{x+y}{xy}\right) + x\frac{df}{dt}\cdot\frac{\partial t}{\partial x}\right]$$

where $t = \frac{x+y}{xy}$

so $\frac{\partial t}{\partial x} = \frac{xy-(x+y)y}{(xy)^2} = \frac{-y^2}{(xy)^2} = -\frac{1}{x^2}$

$$= y\left[f\left(\frac{x+y}{xy}\right) + x\frac{df}{dt}\frac{xy-(x+y)y}{(xy)^2}\right]$$

$$= y\left[f\left(\frac{x+y}{xy}\right) + x\left(\frac{df}{dt}\right)\left(-\frac{1}{x^2}\right)\right]$$

$$= y\left[f\left(\frac{x+y}{xy}\right) - \frac{1}{x}\frac{df}{dt}\right]$$

$$\frac{\partial u}{\partial y} = x\left[f\left(\frac{x+y}{xy}\right) - \frac{1}{y}\frac{df}{dt}\right]$$

← No need to even explicitly compute this since by exchanging $x \leftrightarrow y$; you can see that $g(x,y) = g(y,x)$.

So: $x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y}$

$$= x^2 y \left[f(\cdots) - \frac{1}{x} \frac{df}{dt} \right] - y^2 x \left[f(\cdots) - \frac{1}{y} \frac{df}{dt} \right]$$

$$= xy \left\{ \left[x f(\cdots) - \frac{df}{dt} \right] - y f(\cdots) + \frac{df}{dt} \right\}$$

cancel.

$$= xy \, (x-y) f\left(\frac{x+y}{xy} \right)$$

$$= (x-y) \Big[\underbrace{xy f\left(\frac{x+y}{xy} \right)}_{= u(x,y)} \Big]$$

$$= \boxed{(x-y) \, u} \implies \boxed{G(x,y) = x-y}$$

$\boxtimes$