MITES 2009 : Calculus II : Problem set 1

Due : June 30, 09 [Tues]: 9 AM to Nick

Problems

1.) Let $f(x) = \frac{1}{1+x}$. What is

(i) $f(f(x))$ and for which x does this make sense?

(ii) $f(1/x)$

(iii) $f(cx)$

(iv) $f(x+y)$

(v) $f(x)+f(y)$

(vi) For which numbers "C" is there a number x such that $f(cx)=f(x)$

(Hint: There are a lot more than you might think at first glance

(vii) For which numbers c is it true that $f(cx)=f(x)$ for two different numbers x?

2.) Let $g(x)=x^2$, and let

$$h(x) = \begin{cases} 0 , & x \text{ rational} \\ 1 , & x \text{ irrational} \end{cases}$$

(i) For which y is $h(y) \leq y$?

(ii) For which y is $h(y) \leq g(y)$?

(iii) What is $g(h(z)) - h(z)$?

(iv) For which w is $g(w) \leq w$?

(v) For which ε is $g(g(\varepsilon)) = g(\epsilon)$?

3.) Sketch the following functions. [No calculators allowed!!]

(i) $f(x) = x + \frac{1}{x}$

(To help you with your sketch: What happens for x near 0, and for large x? Where does the graph lie in relation to the graph of the identity function ($g(x) = x$)?
Why does it suffice to consider only positive x at first?)

(ii) $f(x) = x - 1/x$

(iii) $f(x) = x^2 - 1/x^2$

(iv) Graph $f(x) = |\sin x|$.

(v) Graph $f(x) = \sin^2 x$

Sketching these two, you'll notice that there's an important difference between them.

(vi). Sketch the relationship between x & y that satisfies $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(vii) Sketch the relationship between x & y that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

a & b are constants.

4.) Find the following limits: (after some algebraic manipulations, you can do these)

(i) $\lim_{x \to 1} \frac{x^2 - 1}{x + 1}$

(ii) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

(iii) $\lim_{x \to y} \frac{x^n - y^n}{x - y}$ (n is some positive integer)

(iv) $\lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$

5) Evaluate the following limits in terms of the number

$$\alpha = \lim_{x\to 0} \frac{\sin x}{x}$$

(i.e., leave your answer in terms of α.)

You can assume that $\alpha \neq 0$.

(you'll figure out later on what it actually is.)

(i) $\lim_{x\to 0} \frac{\sin 2x}{x}$

(ii) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

(iii) $\lim_{x\to 0} \frac{\sin^2 2x}{x}$

(iv) $\lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

(v) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

(vi) $\lim_{x\to 0} \frac{\tan^2 x + 2x}{x + x^2}$

(vii) $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

(viii) $\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

(ix) $\lim_{x\to 1} (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3$

Now, can you justify, using our discussion of Taylor series in class, why $\alpha = 1$?

(Hint! Think what happens to Taylor series of $\sin x$ for $x \approx 0$.)

6.) In each of the following cases, find a δ such that

$|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x-a| < \delta$.

(i) $f(x) = x^4$; $l = a^4$

(ii) $f(x) = \frac{1}{x}$; $a=1, l=1$

(iii) $f(x) = \sqrt{x}$; $a=1, l=1$

7.) What is l'Hôpital's rule and why is it true?
Justify carefully, step-by-step, your reasoning.

8) Find $f'(x)$ [i.e., $\frac{df}{dx}$]:

(i) $f(x) = \sin(x + x^2)$

(ii) $f(x) = \sin\left(\frac{\cos x}{x}\right)$

(iii) $f(x) = \sin^3(\sin^2(\sin x))$

(iv) $f(x) = \frac{1}{1+x}$

9.) Find f' in terms of g' if: (Note: a is constant)

(i) $f(x) = g(x + g(a))$

(ii) $f(x) = g(x \cdot g(a))$

(iii) $f(x) = g(x + g(x))$

(iv) $f(x) = g(x)(x-a)$

10.) For each of the following functions, find the maximum and minimum values on the indicated closed intervals.
(Don't forget to check the end points of the interval.)

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$

(ii) $f(x) = \frac{x}{x^2-1}$ on $[0, 5]$

11.) Perform the following integrals. (You don't need any formula sheets. But You'll need to use some "algebraic tricks" to do these integrals. The reason I'm giving you these here is ~~to~~ that these tricks are very useful.)

(i) $\displaystyle\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}}\,dx$ (i.e., $\sqrt[5]{x^3} = (x^3)^{1/5}$, $\sqrt[6]{x} = (x)^{1/6}$)

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1} + \sqrt{x+1}}$

(iii) $\displaystyle\int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}}\,dx$

(iv) $\displaystyle\int \frac{dx}{a^2+x^2}$

12.) Do the following integrations using simple substitutions.

(i) $\int e^x \sin(e^x)\,dx$

(ii) $\int x e^{-x^2}\,dx$

(iii) $\displaystyle\int \frac{\log x}{x}\,dx$ ← (In class, we did this by parts, but try doing substitution this time.)

(iv) $\displaystyle\int \frac{e^x\,dx}{e^{2x} + 2e^x + 1}$

(v) $\int e^{e^x} e^x\,dx$

(vi) $\int \log(\cos x)\tan x\,dx$

13.) Integration by parts

(i) $\int x^2 e^x \, dx$

(ii) $\int x^2 \sin x \, dx$

(iii) $\int (\log x)^3 \, dx$

(iv) $\int \cos(\log x) \, dx$

(v) $\int \sqrt{x} \log x \, dx$

(vi) $\int x (\log x)^2 \, dx$

14) The following integrations can all be done with substitutions of the form $x = \sin u$, $x = \cos u$, etc.

To do some of these, you will need to use:

$$\int \sec x \, dx = \log(\sec x + \tan x)$$

and

$$\int \csc x \, dx = -\log(\csc x + \cot x)$$

You can check that these are indeed true by taking their derivatives.

(It'll also help you to have the derivatives of all trigonometric functions handy)

(i) $\int \frac{dx}{\sqrt{1-x^2}}$ (Hint: let $x = \sin u$)

(ii) $\int \frac{dx}{\sqrt{1+x^2}}$ (Hint: since $\tan^2 u + 1 = \sec^2 u$ you want to use the substitution $x = \tan u$)

(iii) $\int \frac{dx}{\sqrt{x^2-1}}$

(iv) $\int \frac{dx}{x\sqrt{x^2-1}}$

(v) $\int x^3 \sqrt{1-x^2} \, dx$

(vi) $\int \sqrt{1-x^2} \, dx$

You'll also need to use the methods for integrating powers of sin and cos.

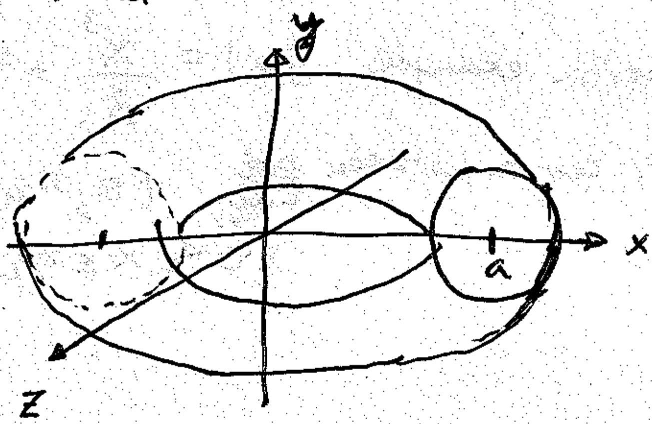
15.) (a) Find the Volume of the solid obtained by revolving the region bounded by the graphs of $f(x) = x$ and $f(x) = x^2$. around the horizontal axis (x-axis)

(Sketch the region as well)

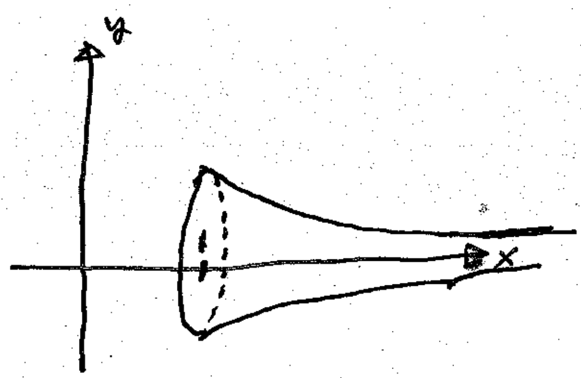
(b) The same but this time, revolved around y-axis.

16) Find Volume of a sphere of radius r by finding a suitable function, then revolving it around x-axis.

17.) Find the volume of the "torus" (donut) obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ $(a>b)$ around the vertical axis.



18.) The graph of $f(x) = \frac{1}{x}$, $x \geq 1$ is revolved around the horizontal axis (x-axis)



(a) Find the volume of the enclosed "infinite trumpet"

(b) show that the surface area is infinite

(c) Suppose that we fill up the trumpet with the finite amount of paint found in (a). It would seem that we have thereby coated the infinite inside surface area

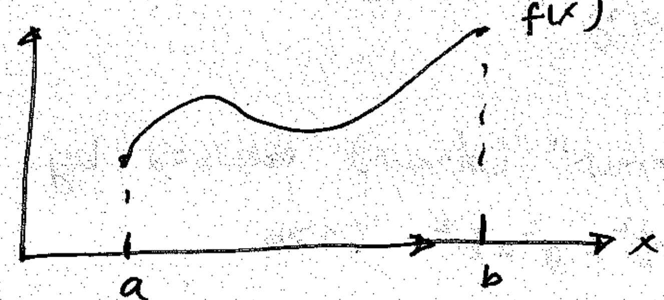
over

→ continued

with only a finite amount of paint!

How is this possible?

19) Derive a formula for computing the length of an arbitrary curve $f(x)$ between $x \in [a, b]$.
(i.e., $a \le x \le b$)



a. Then, apply it to the specific example $f(x) = \frac{x^3}{3} + 47\pi$.

to find the length of this curve between $[0, \pi/4]$
$(0 \le x \le \pi/4)$.

20) Find the Taylor polynomials (of the indicated degree, and at the indicated point) for the following functions.

(i) $f(x) = e^{e^x}$; degree 3, at 0

(ii) $f(x) = e^{\sin x}$; degree 3, at 0

(iii) $\sin(x)$; degree $2n$, at $\pi/2$ (n is (+) integer)

(iv) $\cos(x)$; degree $2n$, at π

(v) e^x ; degree n, at 2

(vi) $f(x) = \dfrac{1}{1+x^2}$; degree $2n+1$, at 0.

21) Using Taylor series, show that when x is close to zero ($x \approx 0$) !

(i) $\sin(x) \approx x$

(ii) $\cos(x) \approx 1 - \dfrac{x^2}{2}$

(iii) $\tan(x) \approx ?$ ← Find to a leading order in x,

(a) Use the Taylor Polynomials of the indicated degree and at the indicated point

for the following functions

(i) $f(x) = e^x$; degree 3, at 0

(ii) $f(x) = e^{2x}$; degree 3, at 0

(iii) [illegible] ; degree 3n, at $\frac{\pi}{2}$ (in [illegible] degree)

(iv) $\ln(x)$; degree 4n, at 1

(v) [illegible] ; degree n, at 0

(vi) $f(x) = \frac{1}{1+x^2}$; degree 2n, at 0

(b) Using Taylor series, show that when x is close to zero (approx.)

(i) $\sin(x) \approx x$

(ii) $\cos(x) \approx 1 - \frac{x^2}{2}$

(iii) $\tan(x) \approx x$ — this is a leading order in x.