MITES 2009 : Calculus II - Problem Set 2.

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(Due on Tues, July 7, 09 at 9 AM.)

For 1 and 2, sketch the given vectors $\vec{v}$ and $\vec{w}$. On your sketch, draw in -$\vec{v}$, $\vec{v}+\vec{w}$, and $\vec{v}-\vec{w}$.

1. $\vec{v}$ =(0, 4) and $\vec{w}$ = (2, -1). 2. $\vec{v}$ = (2, 3, -6) and $\vec{w}$ = (-2, 0, 1).

3. Derive the equation of the line passing through (-1, -1, -1) in the direction of $\hat{y}$.

4. Derive the equation of the line passing through (-5, 0 , 4) and (6, -3, 2).

5. Show that there are no points (x, y, z) satisfying 2x - 3y + z - 2 =0 and lying on the line $\vec{v}$ = (2, -2, -1) + t(1, 1, 1).

6. Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.

7. Using vectors, prove that the medians of a triangle intersect at a point, and this point divides each median in a ratio of 2 : 1.

8. Find a line that lies in the set defined by the equation $x^2+y^2-z^2=1$. How many such lines are there?

For 9 and 10, calculate the dot products.

9. (3, 2, 1)•(1, 2, -1) 10. $\vec{u}\bullet\vec{v}$, where $\vec{u}=\sqrt{3}\hat{x}-315\hat{y}+22\hat{z}$ and $\vec{v}=\dfrac{\vec{u}}{\|\vec{u}\|}$.

11. Find two nonparallel vectors both orthogonal to (1, 1, 1).

12. Find the line through (3, 1, -2) that intersects and is perpendicular to the line $x=-1+t, y=-2+t, z=-1+t$.

13. Find the projection of $\vec{v}=2\hat{x}+\hat{y}-3\hat{z}$ onto $\vec{u}=-\hat{x}+\hat{y}+\hat{z}$.

14. Find an equation for the plane that is perpendicular to $\vec{v}=(1,1,1)$ and passes through $(1,0,0)$.

15. Find an equation for the plane that is perpendicular to the line $\vec{l}(t)=(-1,-2,3)t+(0,7,1)$ and passes through $(2,4,-1)$.

16. Find an equation for the plane that contains the line $\vec{v}=(-1,1,2)+t(3,2,4)$ and is perpendicular to the plane $2x+y-3z+4=0$.

17. Show that adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged; that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+\lambda a_1 & b_2+\lambda b_1 & c_2+\lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In fact, adding a multiple of any row (column) of a matrix to another row (column) leaves the determinant unchanged.

18. Show that two planes given by the equations $Ax+By+Cz+D_1=0$ and $Ax+By+Cz+D_2=0$ are parallel, and that the distance between them is

$$\frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

For 19 and 20, draw the level curves (in the xy - plane) for the given function f and specified values of c. Sketch the graph of $z = f(x, y)$.

19. $f(x, y) = 4 - 3x + 2y$, $c = 0, 1, 2, 3, -1, -2, -3$.   20. $f(x, y) = x^2 + xy$, $c = 0, 1, 2, 3, -1, -2, -3$.

21. Using polar coordinates, describe the level curves of the function defined by

$$\begin{aligned} f(x, y) &= \frac{2xy}{x^2 + y^2} \qquad , \text{ if (x, y)} \neq \text{(0, 0)} \\ f(0, 0) &= 0 \end{aligned}$$

Problems 22 - 26: Compute the following limits, if they exist.

22. $\lim_{(x,y)\to(0,0)} (x^2 + y^2 + 3)$ 23. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2 + 2}$ 24. $\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x + 1}$

25. $\lim_{(x,y)\to(0,0)} \frac{cos(x) - 1 - (x^2/2)}{x^4 + y^4}$ 26. $\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + y^2}$

For 27 - 29, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

27. $f(x, y) = xy$ 28. $f(x, y) = e^{xy}$,   29. $f(x, y) = (x^2 + y^2)log(x^2 + y^2)$.

30. Where does the plane tangent to $z = e^{x-y}$ at $(1, 1, 1)$ meet the z axis?

31. Why should the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ be called "tangent" at $(0, 0)$?

32. A bug finds itself in a toxic environment. The toxicity level is given by $T(x, y) = 2x^2 - 4y^2$. The bug is at $(-1, 2)$. In what direction should it move to lower the toxicity the fastest?

33. (a). Let F be a function of one variable and f a function of two variables. Show that the gradient vector of $g(x, y) = F(f(x, y))$ is parallel to the gradient vector of $f(x, y)$.

(b). Let $f(x, y)$ and $g(x, y)$ be functions such that $\nabla f = \lambda \nabla g$ for some function $\lambda(x, y)$. What is the relation between the level curves of f and g? Explain why there might be a function F such that $g(x, y) = F(f(x, y))$.