MITES 09. : Introduction to Differential Equations

In a high school algebra class, a typical problem you encounter is:

`Solve for x in the following equation:`

$$x^2 + 3x + 2 = 0 \tag{1}$$

Another typical question may look like:

`Solve for y in terms of x in the following equation:`

$$x^2 + y^2 = 1 \tag{2}$$

Both are examples of **algebraic equations**. A question that you may be less familiar with is of the following sort:

`Solve for y in the following equation:`

$$\frac{dy(x)}{dx} + 2 = 0 \tag{3}$$

Above is an example of a **differential equation**: an equation that involves the unknown $y(x)$ and its derivatives. y is a function of x, and the question above is asking us to find a function $y(x)$ whose derivative with respect to x satisfies equation (3). In equation (2), y is a function of x as well, and we recognize it as an equation describing a unit circle (circle with radius 1). Algebraic equations such as (1) and (2) can be solved, in principle, by using algebraic manipulations such as multiplications, divisions, taking square roots, factoring, etc. For example, rearranging equation (2) and taking square root of both sides of the resulting equation yields $y(x) = \pm(1-x^2)^{1/2}$. As for Eq. (1), factoring the equation lets us see that the equation has two solutions: x= -2, and x= -1. But can we use the same techniques to solve for $y(x)$ in the differential equation (3)? At the end of the the day, we would like to say $y(x)$ = some function of x, just as we did when we solved equation (2). But how do we even introduce x into Eq. (3)? As it stands, x doesn't even appear anywhere in Eq. (3). Surely, we need to have it pop out to eventually write down $y(x)$ = something involving x.

There are usually two methods of solving differential equations:

• Method 1: Guess the solution based on your intuition of the physics that's being studied, then check if it's indeed a solution by plugging your guessed function in the equation. For example, if we're studying simple harmonic motion, we would expect that the solution $y(x)$ to be a periodic (oscillating) function such as cos(x) and sin(x). So we would try out $y(x) = cos(x)$ and $y(x) = sin(x)$, plug it into the differential equation, and see if the equation is satisfied.

• Method 2: Systematic way of solving : usually involving integration of the differential equation or some other techniques (e.g. power series expansion of $y(x)$).

Method 2 doesn't always work, mainly because many differential equations are just too difficult to solve by hand (or no systematic method of obtaining exact solution is known for that equation yet). In such cases, we rely on computer algorithms to obtain *approximate* solutions to the equation. Method 1, in principle, should always work since if there is a solution, and you can guess it, then you can check if it's a solution to the equation or not. Of course, the problem is that it's incredibly hard to guess the correct solution without spending days or weeks of trial and error in many situations. Also, there is often more than one solution to an equation, just as Eq. (1) had two solutions. So even if you guessed a correct solution, you need to make sure you haven't left out any other solutions to the equation. For example, say you didn't know how to solve equation (1) systematically (e.g. by factoring, etc.) but instead decided to guess what x should be. And let's say you guessed x= -1, plugged it into Eq. (1), and found that indeed x = -1 satisfied the equation and was thus a solution. Great. How do you know that there aren't any more solutions? After all, x = -2 is also a solution. We will address this important question after some examples.

The subtleties with method 1 aside, we will encounter equations that can usually be solved using both methods. Let's apply the two methods in solving Eq. (3):

Method 1 applied to solving equation (3): Rearranging the equation, we get:

$$\frac{dy(x)}{dx} = -2. \tag{4}$$

Browsing through functions you know in your head, can you think of a function y(x) whose derivative with respect to x gives you -2? How about $y(x) = -2x$? Let's check this guess by substituting it into Eq. (3). This involves taking derivative of -2x, after which Eq. (3) becomes

$$\frac{dy}{dx} = -2$$

Therefore, we found a solution to Eq. (3): $y(x) = -2x$. But are there other solutions to above equation? How about $y(x) = -2x + 1$? Plugging the derivative of this into Eq. (3) lets you see that this is indeed a solution. In fact, we can see that

$$y(x) = -2x + c, \tag{5}$$

where c is any constant (i.e. some number, not a function of x) is a solution to Eq. (3). Thus, there is an infinite amount of solutions to Eq. (3) (one for each value of c), but all of them have the *same form* (Eq. (5)). In fact, there are no other functions that satisfy Eq. (3); Eq. (5) describes all the *class of solutions* to Eq. (3).

Method 2 applied to solving Eq. (3): Stepping aside from our original problem for a bit, suppose you are told the following:

$$\frac{\Delta y}{\Delta x} = -2 \tag{6}$$

This is just telling us that the slope is -2, and by multiplying both sides by Δx, we get

$$\Delta y = -2\Delta x \tag{7}$$

Suppose you are at x= 2. Then you move to x = 5 ($\Delta x = 3$). Then y changes by $-2\Delta x = -6$. But changing from what initial value? If we say $y(x = 2) = c$, then we'd say $y(x = 5) = c - 2\Delta x = c - 6$. Keeping this example in mind, let's go back to our differential equation (3). Just as we did above, we *multiply* both sides of Eq. (3) by dx and get:

$$dy = -2dx \tag{8}$$

Think back to what a derivative means. It's the slope of a tangent line at a particular location on the curve. Thus, what Eq. (8) is saying is that if you're initially standing at $x = x_0$ (the curve has value $y(x_0)$ there), and you move by dx (so you're now at $x_0 + dx$), then the change in value of y is now dy. More specially, $y(x_0 + dx) = y(x_0) + dy$, where $dy = -2dx$ as we found in Eq. (8). Now, say you moved to $x = x_f$ which is quite far away from where you're currently standing ($x = x_0$). But we can always take baby steps (each with *infinitesimal* step size dx) and eventually go from $x = x_0$ to $x = x_f$ after N steps, where $dx = \dfrac{x_f - x_0}{N}$. So we have

$$\begin{aligned}
y(x_f) &= y(x_0) + (-2)dx + (-2)dx + ... + (-2)dx \\
&= y(x_0) + (-2)(dx + dx + ... + dx) \\
&= y(x_0) - 2\sum_{x_0}^{x_f} dx \\
&= y(x_0) - 2Ndx \\
&= y(x_0) - 2(x_f - x_0)
\end{aligned} \tag{9}$$

Now, we can choose x_f to be any position we want (i.e. we can choose our final position x_f to be any value and above procedure would still work). So, instead of writing the subscript "f" in x_f, we drop it and write x (i.e. x_f is an *independent variable*, so we write it as x). Hence we have

$$y(x) = y(x_0) - 2x - 2x_0 \tag{10}$$

Finally, while we vary the final destination x_f to be anything (so we called it x, a variable), we've fixed the initial position x_0. So $y(x_0) - 2x_0$ is a constant, which we can call c. So, writing $c = y(x_0) - 2x_0$, above equation can be rewritten as

$$y(x) = -2x + c \tag{11}$$

This is the solution to the differential equation (3) and matches the one we found by using Method 1 (Eq. (5)).

We have just solved Eq. (3) using two different methods. But let's go back to method 2 for a bit, and in particular, look at Eq. (9). We have been a bit cavalier about writing and summing up dx as if it's nothing special. But in fact, it is very special. By *definition*, it turns out that

$$\sum_{x_0}^{x_f} dx = \int_{x_0}^{x_f} dx \tag{12}$$

Indeed, do the integral and you'll get

$$\begin{aligned} \int_{x_0}^{x_f} dx &= x\Big|_{x_0}^{x_f} \\ &= x_f - x_0 \end{aligned} \tag{13}$$

which is exactly what you get in the last line of Eq. (9). This sum of infinitesimal quantity dx, is called the **Riemann sum**, named after the mathematician Bernhard Riemann (1826-1866) http://en.wikipedia.org/wiki/Riemann, and is THE definition of **integration**. Let's do another example where we really do need to use integration (not a straightforward summation as in Eq. (9)).

Example: Solve the following differential equation:

$$\frac{dy}{dt} - at - v = 0, \tag{14}$$

where a and v are constants. (NOTE: Unless specified otherwise, any letters you see in an equation are constants, not a function).

Solution: In this problem, y is a function of t: $y(t)$. Our goal is to find out what $y(t)$ is. Let's solve this using method 2. Mimicking the previous example, we rearrange the equation to get

$$\frac{dy}{dt} = at + v, \tag{15}$$

Again, mimicking the previous example, we "multiply" both sides by dt and get

$$dy = (at + v)dt \tag{16}$$

Again, we imagine two values of independent variable (in this case, t): $t = t_0$ is our starting position, and $t = t_f$ is our final position. $y(t_0)$ and $y(t_f)$ are the values of y at t_0 and t_f respectively. Then using the same logic as in the previous example, imagine uniformly diving up the interval $[t_0, t_f]$ into a huge number N of segments, each with an infinitesimal length $dt = \dfrac{t_f - t_0}{N}$. Now, if you just write down

$$y(t_f) - y(t_0) = \sum_{t_0}^{t_f} (at + v)dt, \tag{17}$$

how would you evaluate this sum? It is not clear how you would do this off the top of your head because the (at+v) is a function that's *continuously* changing as t is changing. But by making N large, we can make $dt = \frac{t_f - t_0}{N}$ as small as we want (this is what we mean by "infinitesimal" after all).
And if we do so, then within a given infinitesimal interval dt (say in the interval $[t_5, t_5 + dt]$, there isn't much room for the function $at + v$ to change much. That is, the difference between $at_5 + v$ and $a(t_5 + dt) + v$ is infinitesimal:

$$a(t_5 + dt) + v - (at_5 + v) = a(dt) \tag{18}$$

and since dt is infinitesimal, a(dt) is an incredibly small quantity as well (No matter how large a may be, we can always pick N to be sufficiently large enough so that $dt = \frac{t_f - t_0}{N}$ is made small to compensate for the large a, ensuring that the product $a(dt)$ is very small). Then, above sum is accurately approximated by

$$\begin{aligned} y(t_f) - y(t_0) &= \sum_{t_0}^{t_f} (at + v)dt \\ &= (at_0 + v)dt + (at_1 + v)dt + ... + (at_N + v)dt \\ &= \int_{t_0}^{t_f} (at + v)dt \\ &= (\frac{at^2}{2} + vt)\Big|_{t_0}^{t_f} \\ &= \frac{a(t_f^2 - t_0^2)}{2} + v(t_f - t_0) \end{aligned} \tag{19}$$

And as before, by redefining t_f to be t, above equation is rewritten as

$$y(t) = \frac{at^2}{2} + vt + y(t_0) - \frac{at_0^2}{2} - vt_0 \tag{20}$$

And again, mimicking our solution to previous example, we lump together the constant $y(t_0) - \frac{at_0^2}{2} - vt_0$ and write it as c. Hence, above equation (and our solution $y(t)$ to the differential equation (14)) is

$$y(t) = \frac{at^2}{2} + vt + c \tag{21}$$

You'll notice that this is actually an equation familiar to you from kinematics in a first physics course you took. You may have seen it as $x(t) = x_0 + v_0 t + \frac{at_f^2}{2}$, the equation describing position of a particle $x(t)$ with respect to time t, where x_0, v_0, and a are the initial position, velocity, and the constant acceleration respectively. So look at equation (14) again: It is actually describing (as a function of time t), the position $y(t)$ of a particle with an initial veloctiy v and a constant accleration a.

Example: Solve the following differential equation:

$$\frac{dy}{dt} - f(t) = 0 \tag{22}$$

Solution: Applying method 2 to this (and mimicking the previous two examples), we get

$$
\begin{aligned}
\frac{dy}{dt} &= f(t), \\
\Rightarrow dy &= f(t)dt, \\
\Rightarrow \int_0^{y_f} dy &= \int_0^{t_f} f(t)dt \\
\Rightarrow y(t_f) - y(0) &= \int_0^{t_f} f(t)dt \\
\Rightarrow y(t_f) &= \int_0^{t_f} f(t)dt + y(0)
\end{aligned}
\tag{23}
$$

Notice that here, we just picked $t_0 = 0$. This is fine; you can pick the starting position t_0 to be anything you want. Often $t_0 = 0$ is the easiest choice (since then many terms usually become zero, making us write less). Since we don't want to keep writing the subscript "f" in t_f (which is an independent variable anyway), we drop the subscript and write

$$y(t) = y(0) + \int_0^t f(t')dt' \tag{24}$$

This is the general solution to the differential equation. Since we don't know the actual function $f(t)$, we can't go ahead and finish off the integral, so this is as far as we can go. In fact, for many of the equations one encounters, the $f(t')$ is *non-analytic*, meaning that we can't do the integral by hand and require computer algorithms. Also, we picked t=0 as a reference point in doing the integration above, but we can always pick another point, say $t = 2.5$, or $t = \pi$ and many other possible choices. So, we could've written it as

$$y(t) = c + \int_a^t f(t')dt', \tag{25}$$

where $y(a) = c$. Where you pick the reference point a (and thus $y(a) = c$) is completely arbitrary.

Problem Set 3 for Calculus II: Solve the following differential equations. For these problems, use method 2, but you may also try "method 1" if you can guess the right solution (you'll see that this guessing game is not so easy. But guessing is easier if the equations are based on some physical phenomena that you're modeling). Remember that the key in solving these equations (using "method 2") is isolating the two variables by themselves (one variable on the left hand side, the other variable on the right hand side of the equal sign) *before* integrating the equation.

1. $\frac{dz}{dx} + 5x = 0$
2. $\frac{dy}{dt} + at^2 + b = 0$
3. $\frac{dx}{dz} + e^z = z$ Hint: $\int e^t dt = e^t + c$
4. $\frac{dx}{dt} + \frac{1}{t} = 3$ Hint: $\int \frac{1}{y} dy = ln(y) + c$
5. $x\frac{dx}{dt} + bt = 0$
6. $\frac{dk}{dt} + k = 3$

(26)

For the last question, the following properties of exponential e and natural logarithm ln should help you out.

Properties of the natural logarithm "ln(x)" and the exponential "e^x" (also written as "exp(x)") :

(i) $ln(xy) = ln(x) + ln(y)$

(ii) $ln(x/y) = ln(x) - ln(y)$

(iii) $ln(y^n) = nln(y)$ This follows from (i) and (ii). Can you prove this relation using (i) and (ii)?

(iv) $e^x e^y = e^{x+y}$

(v) $exp(ln(x)) = x$ and $ln(exp(x)) = x$. (i.e. exp(x) and ln(x) are inverse function of each other.)