MITES 2009 : Calculus II - Problem Set 4.

Massachusetts Institute of Technology
Instructor: Hyun Youk
TA: Nicholas Villalva
(Due on Tues, July 21, 09 at 9 AM.)

Problem 1. Solving differential equations I: Consider the differential equation

$$y'' - 5y' + 6y = 0. \tag{1}$$

Here, $y' = \frac{dy}{dx}$ and thus $y'' = \frac{d^2y}{dx^2}$. $y = y(x)$ (function of x).

(a) Verify that $y = e^{2x}$ and $y = e^{3x}$ are both solutions.
(b) Verify that $y = c_1 e^{2x} + c_2 e^{3x}$ is a solution for *every* choice of the constants c_1 and c_2.
(c) Show that above differential equation (1) is a *linear* equation.
(Hint: Consider $y_1(x)$ and $y_2(x)$ - two solutions of above equation. What can you say about $y_1(x) + y_2(x)$? It will help you to review what the definition of linear differential equation is).

Problem 2. Solving differential equations II: For what values of the constant m will $y = e^{mx}$ be a solution of the differential equation

$$2y''' + y'' - 5y' + 2y = 0? \tag{2}$$

Use the ideas in problem 1 to find a solution containing three arbitrary constants c_1, c_2, c_3. Could you have more arbitrary constants (say c_4) in your solution? Why or why not?

Problem 3. Modeling real world using ODE I: Consider a column of air of cross-sectional area 1 square inch extending from sea level up to "infinity." The atmospheric pressure p at an altitude h above sea level is the weight of the air in this column above the altitude h. Assuming that the density of the air is proportional to the pressure, show that p satisfies the differential equation

$$\frac{dp}{dh} = -cp, \qquad c > 0 \tag{3}$$

and obtain the formula $p = p_0 e^{-ch}$, where p_0 is the atmospheric pressure at sea level.

Problem 4. Modeling real world using ODE II: If m is the mass and $y(t)$ is the height of the particle above ground at time t, then Newton's second law states

$$\frac{d^2y}{dt^2} = -g. \tag{4}$$

(a) What is the velocity of the particle at time t, $v(t) = \frac{dy(t)}{dt}$?
(b) What is the height of the particle at time t, $y(t)$?
(c) If air resistance acting on a falling body of mass m exerts a retarding force proportional to the square of the velocity, then Eqn. (4) becomes

$$\frac{dv}{dt} = -g - cv^2, \tag{5}$$

where $c = \frac{k}{m}$. If $v = 0$ when $t = 0$, find v as a function of t. What is the velocity of the particle $v(t)$? What is $v(t)$ after some very long time? This velocity attained, after long time, is called the *terminal velocity*.

Problem 5. Drilling a hole through Earth: Inside the earth, the force of gravity is proportional to the distance from the center. If a hole is drilled through the earth from pole to pole, and a rock is dropped into the hole, with what velocity will it reach the center?

Problem 6. Solving 2nd order differential equations:
(a) By guessing $y(x) = e^{\alpha x}$ (where you need to find what α is), find the most *general* solution to the differential equation

$$y'' - y' - 2y = 0 \tag{6}$$

(Hint: How many arbitrary (in class we called this initial condition dependent) constants should exist in the general solution?)
(b). Find a and b so that $y_p = ax + b$ is a *particular* solution of the complete equation

$$y'' - y' - 2y = 4x \tag{7}$$

Use this solution and the result of part (a) to write down the general solution of this equation.

Problem 7. Solving 2nd order differential equations II: Use inspection (insightful guess) or experiment (trial and error) to find a particular solution for each of the following equations:

(a) $x^3y'' + x^2y' + xy = 1$
(b) $y'' - 2y' = 6$
(c) $y'' - 2y = sin(x)$

Also state for each equation, if the equation is linear or not.

Problem 8. Solving 2nd order differential equations III: Consider the differential equation

$$y'' + a^2y = sin(bx) \tag{8}$$

where $y = y(x)$.

(a) What is the solution to this differential equation? Remember, the most general solution should contain the right number of arbitrary (initial condition-dependent) constants. Here, a and b are positive constants.

(b) Without too much work, and applying "linearity" of above differential equation, write down the most general solution to the following equation:

$$y'' + a^2y = 3sin(2x) + 5sin(10x) + 90sin(3x) + 100sin(101x) \tag{9}$$

Be sure to justify how you got your solution.
(Hint: Breaking up above equation into "shorter" equations will help.)

(c) Again, without too much work, applying "lineartiy" of above differential equation, write down the most general solution to the following equation:

$$y'' + a^2y = \sum_{n=1}^{\infty} b_n sin(nx), \tag{10}$$

where b_n is a constant for each n.

(d) What would the most general solution to the following equation be?

$$y'' + a^2 y = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n cos(nx) + b_n sin(nx)), \tag{11}$$

where a_n and b_n are constants, as is a_0. Notice that the right hand side is the Fourier series. You have just derived how to find the solution to any function $f(x)$ (whose Fourier series represented on the right hand side of above equation). Notice that we could have cooked up a different equation, so that the left hand side of "=" looks different, while the right hand side of "=" is still the Fourier series. As long as the equation is linear, the formalism you have just derived in this problem using linearity is valid. This means that you have just found out how to solve a large body of *linear* differential equations, not just of 2nd order linear equation.

Problem 9. Fourier series: Here we prove that

$$\int_{-\pi}^{\pi} sin(nx)cos(mx) = 0, \qquad \text{(for any integers } n \text{ and } m\text{)}$$
$$\int_{-\pi}^{\pi} sin(nx)sin(mx) = 0, \qquad \text{(if } n \neq m\text{)}$$
$$\int_{-\pi}^{\pi} cos(nx)cos(mx) = 0, \qquad \text{(if } n \neq m\text{)}$$

Here's how the proof works. I will guide you step by step.

(a) Let $u(x) = sin(nx)$ and $v(x) = cos(mx)$. First show that

$$u'' + n^2 u = 0$$
$$v'' + m^2 v = 0$$

(b) Using integration by parts, show that

$$\int_{-\pi}^{\pi} u''(x)v(x)dx = -\int_{-\pi}^{\pi} u'(x)v'(x)dx$$
$$\int_{-\pi}^{\pi} u(x)v''(x)dx = -\int_{-\pi}^{\pi} u'(x)v'(x)dx$$

(c) This time, using the differential equations you derived in (a), show that

$$\int_{-\pi}^{\pi} u''(x)v(x)dx = -n^2 \int_{-\pi}^{\pi} u(x)v(x)$$
$$\int_{-\pi}^{\pi} u(x)v''(x)dx = -m^2 \int_{-\pi}^{\pi} u(x)v(x)$$

(d) Using (b) and (c), show that

$$0 = (m^2 - n^2) \int_{-\pi}^{\pi} u(x)v(x)dx$$

Why does this prove that $\int_{-\pi}^{\pi} sin(nx)cos(mx)dx = 0$ if $n \neq m$? Can you show, without doing any integration (using parity of the integrand), why $\int_{-\pi}^{\pi} sin(nx)cos(nx)dx = 0$?

(e) Show that when $n \neq m$, then $\int_{-\pi}^{\pi} sin(nx)sin(mx)dx = 0$ and $\int_{-\pi}^{\pi} cos(nx)cos(mx)dx = 0$. To do this, do not try to evaluate the integrals right away. You can use a sum or difference of $cos((n+m)x)$ and $cos((n-m)x)$ to re-express the product of sines and the product of cosines. Then you can use the parity of the resulting integrand to justify why the integrals are zero.

(f) Finally, show, by doing a simple one-liner integral, that

$$\int_{-\pi}^{\pi} sin^2(nx)dx = \int_{-\pi}^{\pi} cos^2(nx)dx = \pi \tag{12}$$

Problem 10. Fourier series

(a) Find the Fourier series of the function defined by

$$\begin{aligned} f(x) &= -\frac{\pi}{2} - \frac{1}{2}x, \qquad && -\pi \leq x < 0 \\ f(x) &= \frac{\pi}{2} - \frac{1}{2}x, \qquad && 0 \leq x \leq \pi \end{aligned}$$

First, sketch f(x) so that you see what the function you're trying to represent by its Fourier series looks like.

(b) Find the Fourier series of the function defined by

$$\begin{aligned} f(x) &= \pi, \qquad && -\pi \leq x \leq \frac{\pi}{2}; \\ f(x) &= 0, \qquad && \frac{\pi}{2} < x \leq \pi. \end{aligned}$$

Again, sketch f(x) before finding the Fourier series.

(c) Find the Fourier series of the function

$$f(x) = 53sin(2x) + 35cos(100x) + \pi sin(35x) + 2\pi cos(77x)$$

This is a trick question. If you find yourself doing any integrals, stop and think. It helps you to look at the relationships you proved in problem 9.