6.) Final Exam (Practice) Solutions to Probs 6 ~ 9

7/29/09
(Calc II)

1(a) $y'' + y' + y = 3x$

$$\left.\begin{aligned} &y_G'' + y_G' + y_G = 0 \quad \ldots (1) \\ &y_p'' + y_p' + y_p = 3x \quad \ldots (2) \end{aligned}\right\} \text{Break into 2 eqns.}$$

- Solve eqn (2): Guess $y_p(x) = ax + b$. (a & b to be determined)

Check: $y_p'' + y_p' + y_p = 3x$ ← Need this to be satisfied

$\Rightarrow a + ax + b = 3x$

$\Rightarrow ax + (a+b) = 3x \Rightarrow \boxed{y_p(x) = 3x - 3}$ ← particular solution

$\boxed{a = 3}$, $a + b = 0 \Rightarrow \boxed{b = -3}$

- Solve eqn (1): $y_G'' + y_G' + y_G = 0$

Guess: $y_G(x) = Ce^{mx}$ C, m ← constants.

Check: $\underbrace{Ce^{mx}}_{\neq 0}\underbrace{\left[m^2 + m + 1\right]}_{= 0} = 0 \Rightarrow m^2 + m + 1 = 0$

$\Rightarrow m_\pm = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$= \dfrac{-1 \pm i\sqrt{3}}{2}$

Hence: $\boxed{y_G(x) = c_1 e^{m_+ x} + c_2 e^{m_- x}}$

← solution to eq'n (1) (c_1, c_2 are arbitrary constants)

- Finally, the general sol'n is:

$$\boxed{\begin{aligned} y(x) &= y_G(x) + y_p(x) \\ &= c_1 e^{m_+ x} + C_2 e^{m_- x} + 3x - 3 \end{aligned}}$$

$\boxed{\text{b.)}}$ $y'' + 3y = \sin x$

↳ $y_G'' + 3y_G = 0$ ... ①

$y_p'' + 3y_p = \sin x$ ... ②

Solve ② : Guess $y_p(x) = A\sin x$

Check : $y_p'' + 3y_p = \sin x$

$\Rightarrow -A\cancel{\sin x} + 3A\cancel{\sin x} = \cancel{\sin x} \Rightarrow 2A = 1 \Rightarrow \boxed{A = 1/2}$

So : $\boxed{y_p(x) = \frac{\sin x}{2}}$ ← particular sol'n

Solve ① : Guess $y_G(x) = ce^{mx}$

Check : $y_G'' + 3y_G = 0$

$\Rightarrow \cancel{c}m^2\cancel{e^{mx}} + 3\cancel{c}\cancel{e^{mx}} = 0 \Rightarrow m^2 + 3 = 0 \Rightarrow \boxed{m_\pm = \pm\sqrt{3}i}$

$\therefore \boxed{y_G(x) = c_1 e^{m_+ x} + c_2 e^{m_- x}}$

(C_1, c_2 : arbitrary constants)

∴ The general solution :

↳ $y(x) = y_G(x) + y_p(x)$

$= \boxed{c_1 e^{\sqrt{3}ix} + c_2 e^{-\sqrt{3}ix} + \frac{\sin x}{2}}$

$\boxed{c.)}$ $y' + ay = 0.$

Guess: $ce^{mx} = y(x)$

$\Rightarrow$ check: $m + a = 0 \Rightarrow \boxed{m = -a}$ $\Rightarrow \boxed{y(x) = ce^{-ax}}$ ← The general solution

so: If $y(0) = 0$, then $y(0) = ce^{-a\cdot 0} = c$

$\Rightarrow \boxed{0 = c}$ so $\boxed{y(x) = 0}$

(solution if $y(0) = 0$)

$\boxed{1(d)}$ $y' = g - ay^2$ ($g, a > 0$, constants.)

$\Rightarrow$

$$\frac{dy}{dx} = g - ay^2$$

"systematic method of solving ("Method 2" in pset 3.)

$$\Rightarrow \int_{y(0)}^{y(x)} \frac{dy}{g - ay^2} = \int_0^x dx$$

$$\Rightarrow x = \frac{1}{g}\int_{y(0)}^{y(x)} \frac{dy}{1 - \left(\frac{a}{g}\right)y^2}$$

Let $u = \sqrt{\frac{a}{g}}\, y \Rightarrow du = \sqrt{\frac{a}{g}}\, dy$

Let $u_0 = \sqrt{\frac{a}{g}}\, y(0)$

$$= \frac{1}{g}\int_{u_0}^{u} \frac{du\sqrt{\frac{g}{a}}}{1 - u^2}$$

$$= \frac{1}{\sqrt{ga}}\int_{u_0}^{u} \frac{du}{1 - u^2}$$

let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta$

$$= \frac{1}{\sqrt{ga}}\int_{\theta_0}^{\theta} \frac{-\cancel{\sin\theta}\, d\theta}{\sin^{\cancel{2}}\theta}$$

$$= \frac{-1}{\sqrt{ga}}\int_{\theta_0}^{\theta} d\theta \csc\theta$$

To do this integral: recall from pset 1

$$= \frac{+1}{\sqrt{ga}} \ln(\csc\theta + \cot\theta)\Big|_{\theta_0}^{\theta}$$

$\theta = \mathrm{ArcCos}(u)$
$= \mathrm{ArcCos}\left(\sqrt{\frac{a}{g}}\, y(x)\right)$

$$\Rightarrow x + \frac{\ln[\csc\theta_0 + \cot\theta_0]}{\sqrt{ga}} = \frac{1}{\sqrt{ga}} \ln\left[\csc\left[\mathrm{ArcCos}\left(\sqrt{\frac{a}{g}}\, y(x)\right)\right] + \cot\left[\mathrm{ArcCos}\left(\sqrt{\frac{a}{g}}\, y(x)\right)\right]\right]$$

$$\Rightarrow \boxed{\sqrt{ga}\,x + \ln\left[\csc\theta_0 + \cot\theta_0\right] = \ln\left[\csc\left[\mathrm{ArcCos}\left(\sqrt{\tfrac{a}{g}}\,y(x)\right)\right] + \cot\left[\mathrm{ArcCos}\left(\sqrt{\tfrac{a}{g}}\,y(x)\right)\right]\right]}$$

← Note: With more work, we can solve for $y(x)$ here. But leaving the answer in this form is fine for our course.

1(e) $2y''' + y'' - 5y' + 2y = 0$

Guess: $y(x) = Ce^{mx}$

check: $\underbrace{[2m^3 + m^2 - 5m + 2]}_{=0}\,\underbrace{Ce^{mx}}_{\neq 0} = 0$

$\Rightarrow \quad 2m^3 + m^2 - 5m + 2 = 0$ → Leads to 3 possible values of m.

To find these 3 values, first guess one of them, by starting with simple #'s such as ($m=0$, $m=1$, $m=-1$, and so on.)

Guess: $m = 1$, then $2 + 1 - 5\cdot(1) + 2 = 0$ ✓ works.

$\therefore\ 0 = 2m^3 + m^2 - 5m + 2$

$= (m-1)(am^2 + bm + c)$ ← Now, need to find a, b, and c values. To do so, just expand out and check.

$\Rightarrow \quad am^3 = 2m^3 \Rightarrow a = 2$

$bm^2 - am^2 = 1m^2 \Rightarrow b = 3$

$-c = 2 \Rightarrow c = -2$

$\therefore\ 0 = \underbrace{(m-1)}_{=0}\,\underbrace{(2m^2 + 3m - 2)}_{=0}$

$(m-1) = 0$ when $m = 1$ ← found above

$(2m^2+3m-2) = 0$ ← when $m = \dfrac{-3 \pm \sqrt{9 - 4(2)(-2)}}{4}$ ← Get 2 more values of m from here.

$= \dfrac{-3 \pm 5}{4} \Rightarrow m_+ = \dfrac{1}{2}$, $m_- = -2$

Hence, 3 values of m ; $\boxed{m_1 = 1, \quad m_2 = \frac{1}{2}, \quad m_3 = -2}$

$$\therefore \boxed{y(x) = c_1 e^{x} + c_2 e^{x/2} + c_3 e^{-2x}} \leftarrow \text{General solution}$$

(f) $y'' - 5y' + 6y = 0$

Guess: $y(x) = c e^{mx}$

Check: $\underbrace{[m^2 - 5m + 6]}_{=0} \underbrace{c e^{mx}}_{\neq 0} = 0$

$$\Rightarrow m^2 - 5m + 6 = 0 \Rightarrow m_\pm = \frac{5 \pm \sqrt{25 - 4(6)}}{2} \Rightarrow m_+ = 3, \; m_- = 2$$

$$\therefore \boxed{y(x) = c_1 e^{3x} + c_2 e^{2x}} \leftarrow \text{The general solution}$$

(g) $y'' + a^2 y = 3\sin(10x) + 100\sin(3x) + 2\cos(9x) + 27\cos(74x)$

To solve this, ~~note that~~ first we solve:

$$y_p'' + a^2 y_p = C\sin(bx)$$

~~Solve~~ Guess: $y_p(x) = A\sin(bx)$

Check: $[-b^2 A + a^2 A]\cancel{\sin(bx)} = C\cancel{\sin(bx)}$

$$\Rightarrow A = \frac{C}{a^2 - b^2}$$

So: breaking up our original differential equation into 5 separate eqns:

$$y_G'' + a^2 y_G = 0 \longrightarrow y_G(x) = C_1 e^{iax} + C_2 e^{-iax}$$ [C_1 & C_2 : arbitrary constants]

$$y_p'' + a^2 y_p = 3\sin(10x) \longrightarrow y_p(x) = \frac{3}{a^2 - 100}\sin(10x)$$

$$y_p'' + a^2 y_p = 100\sin(3x) \longrightarrow y_p(x) = \frac{100}{a^2 - 9}\sin(3x)$$

$$y_p'' + a^2 y_p = 2\cos(9x) \longrightarrow y_p(x) = \frac{2}{a^2 - 81}\cos(9x)$$

$$y_p'' + a^2 y_p = 27\cos(74x) \longrightarrow y_p(x) = \frac{27}{a^2 - (74)^2}\cos(74x)$$

The general solution is all 5 soln's added together :

$$\boxed{y(x) = C_1 e^{iax} + C_2 e^{-iax} + \frac{3}{a^2-100}\sin(10x) + \frac{100}{a^2-9}\sin(3x) + \frac{2}{a^2-81}\cos(ax) + \frac{27}{a^2-(74)^2}\cos(74x)}$$

(h). $y'' + c^2 y = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$

First solve :

- $y_p'' + c^2 y_p = \frac{a_0}{2}$

Guess: $y_p(x) = A$.

$\Rightarrow \quad c^2 A = \frac{a_0}{2} \Rightarrow \boxed{A = \frac{a_0}{2c^2}} \Rightarrow y_p(x) = \frac{a_0}{2c^2}$ ← particular soln.

∴ The general solution : (Add up all solutions: Using above result, and the result from (g))

$$\boxed{y(x) = c_1 e^{icx} + c_2 e^{-icx} + \frac{a_0}{2c^2} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{c^2-n^2}\cos(nx) + \frac{b_n}{c^2-n^2}\sin(nx)\right\}}$$

7.) (a) $\int_{y=0}^{1}\int_{x=0}^{1}(x^3+y^2)\,dx\,dy$

$= \int_{y=0}^{1}\left\{\frac{x^4}{4} + xy^2\right\}\Big|_{x=0}^{1} dy$

$= \int_{y=0}^{1}\left\{\frac{1}{4} + y^2\right\} dy$

$= \left[\frac{y}{4} + \frac{y^3}{3}\right]\Big|_{y=0}^{1}$

$= \left[\frac{1}{4} + \frac{1}{3}\right] = \boxed{\frac{7}{12}}$

(b) $\int_{y=0}^{1}\int_{x=0}^{1} y e^{xy}\,dx\,dy$

$= \int_{y=0}^{1}\left\{e^{xy}\right\}\Big|_{x=0}^{1} dy$

$= \int_{y=0}^{1}\left\{e^{y} - 1\right\} dy$

$= \left[e^{y} - y\right]\Big|_{y=0}^{1}$

$= e^1 - 1 - 1$

$= \boxed{e-2}$

(c) $$\int_{y=0}^{1}\int_{x=0}^{1} (xy)^2 \cos(x^3)\,dx\,dy$$

$$= \int_{y=0}^{1} \left\{ \frac{y^2 \sin(x^3)}{3} \right\}\Bigg|_{x=0}^{1} dy$$

$$= \int_{y=0}^{1} \left\{ \frac{y^2 \sin(1)}{3} \right\} dy$$

$$= \frac{\sin(1)}{3}\; \frac{y^3}{9}\Bigg|_{y=0}^{1}$$

$$= \boxed{\frac{\sin(1)}{27}}$$

(d) $$\int_{y=0}^{1}\int_{x=0}^{1} (x^2 + 2xy + y\sqrt{x})\,dx\,dy$$

$$= \int_{y=0}^{1} \left\{ \frac{x^3}{3} + x^2 y + \frac{2y\,x^{3/2}}{3} \right\}\Bigg|_{y=0}^{1}$$

$$= \int_{y=0}^{1} \left\{ \frac{1}{3} + y + \frac{2y}{3} \right\} dy$$

$$= \left[\frac{y}{3} + \frac{y^2}{2} + \frac{y^2}{3} \right]\Bigg|_{y=0}^{1}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{2}{3} + \frac{1}{2} = \boxed{\frac{7}{6}}$$

(e) $$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} x^2\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left\{ \frac{x^3}{3} \right\}\Bigg|_{x=0}^{1} dy\,dz$$

$$= \frac{1}{3}\int_{z=0}^{1} y\Big|_{0}^{1} dz$$

$$= \frac{1}{3}\int_{z=0}^{1} dz$$

$$= \frac{1}{3}\, z\Big|_0^1$$

$$= \boxed{\frac{1}{3}}$$

(f) $$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} e^{-xy}\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \frac{-e^{-xy}}{y}\Bigg|_{x=0}^{1} dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[-\frac{e^{-y}}{y} + \frac{1}{y} \right] dy\,dz$$

$$= \int_{y=0}^{1} \left[\frac{-e^{-y}}{y} + \frac{1}{y} \right] dy$$

$$= \int_{y=0}^{1} \left[\frac{1 - e^{-y}}{y} \right] dy$$

To do this integral, we can use Taylor series of e^{-y}:

$$e^{-y} = 1 - y + \frac{y^2}{2} - \frac{y^3}{3!} + \frac{y^4}{4!} - \cdots$$

(and so on)
(Infinite sum)

$$= \int_{y=0}^{1} \left[\frac{1 - \left(1 - y + \frac{y^2}{2} - \frac{y^3}{3!} + \frac{y^4}{4!} - \cdots\right)}{y} \right] dy$$

$$= \int_{y=0}^{1} \frac{\left(y - \frac{y^2}{2} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots\right)}{y}\, dy$$

over

$$= \int_{y=0}^{1} \left\{ 1 - \frac{y}{2} + \frac{y^2}{3!} - \frac{y^3}{4!} + \dots \right\} dy$$

$$= \left\{ y - \frac{y^2}{2 \cdot 2} + \frac{y^3}{3 \cdot 3!} - \frac{y^4}{4 \cdot 4!} + \dots \right\} \Bigg|_{y=0}^{1}$$

$$= 1 - \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3!} - \frac{1}{4 \cdot 4!} + \frac{1}{5 \cdot 5!} \dots$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n)(n!)}}$$

1(g)
$$\int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=x^2+y^2}^{x+y} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{2x} \{ z \} \Big|_{z=x^2+y^2}^{x+y} dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{2x} \{ x + y - x^2 - y^2 \} dy\, dx$$

$$= \int_{x=0}^{1} \left\{ xy + \frac{y^2}{2} - x^2 y - \frac{y^3}{3} \right\} \Bigg|_{y=0}^{2x} dx$$

$$= \int_{x=0}^{1} \left\{ 2x^2 + \frac{4x^2}{2} - 2x^3 - \frac{8x^3}{3} \right\} dx$$

$$= \left[\frac{2x^3}{3} - \frac{2x^3}{3} - \frac{2x^4}{4} - \frac{8x^4}{12} \right] \Bigg|_{x=0}^{1}$$

$$= -\frac{2}{4} - \frac{8}{12}$$

$$= -\frac{14}{12} = \boxed{-\frac{7}{6}}$$

8.) $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ ← If 3-dimensional vector field

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ ⟵ If 2-dimensional vector field

(a)

$\nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (-y, x)$ "dot product"

$= \underbrace{\frac{-\partial y}{\partial x}}_{=0} + \underbrace{\frac{\partial x}{\partial y}}_{=0} = \boxed{0}$

(b) $\nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(e^{xy}, -e^{xy}, e^{yz}\right)$

$= \frac{\partial e^{xy}}{\partial x} + \frac{\partial}{\partial y}(-e^{xy}) + \frac{\partial}{\partial z}(e^{yz})$

$= ye^{xy} - xe^{xy} + ye^{yz}$

$= \boxed{(y-x)e^{xy} + ye^{yz}}$

(c) $\nabla \cdot \vec{V} = \underbrace{\frac{\partial (yz)}{\partial x}}_{=0} + \underbrace{\frac{\partial (xz)}{\partial y}}_{=0} + \underbrace{\frac{\partial (xy)}{\partial z}}_{=0}$

$= \boxed{0}$

(d) $\nabla \cdot \vec{V} = \frac{\partial (x)}{\partial x} + \frac{\partial}{\partial y}(y + \cos(x)) + \frac{\partial}{\partial z}(z + e^{xy})$

$= 1 + 1 + 1$

$= \boxed{3}$

(e) $\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}[(x+y)^2] + \frac{\partial}{\partial z}[(x+y+z)^2]$

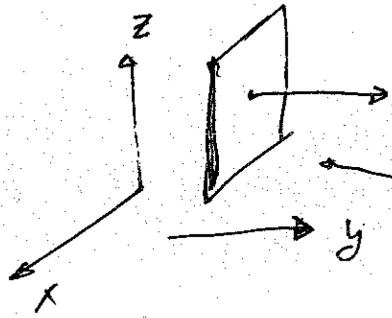
$= \boxed{2x + 2(x+y) + 2(x+y+z)}$

(f) $\boxed{\vec{V}(x,y) = (1,1,0)}$

(g) $\boxed{\vec{V}(x,y) = (1,0,\pi)}$

(h) $\vec{F}(x,y,z) = (2x, 5y, 3z)$

$\{(x,y,z) \mid 2 \le x \le 5,\ y=0,\ -5 \le z \le 5\}$



$\hat{y} = \hat{n} = (0,1,0)$

on this rectangle face: $y=0$

$\Rightarrow \vec{F}(x,0,z) = (2x, 0, 3z).$

$\vec{F} \cdot \hat{n} = \vec{F} \cdot \hat{y}$

$= (2x, 0, 3z) \cdot (0,1,0)$

$= 0.$

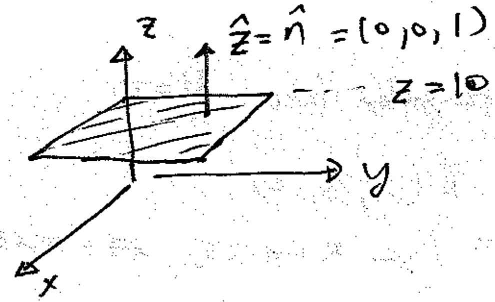
Total flux

$\therefore \Phi_{total} = \int_{\text{rectangular face}} \vec{F} \cdot \hat{n}\, da = \int 0 \cdot da$

$= \boxed{0.}$ (No flux)

$\boxed{(i)}$ $\vec{V}(x,y,z) = (x-y,\ xyz,\ e^{x+z^2})$

$\{(x,y,z):\ 2 \le x \le 5,\ 2 \le y \le \pi,\ z = 10\}$

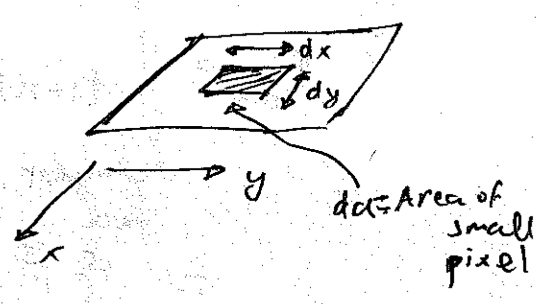


On the rectangular face,

$$\vec{V}(x,y,10) = (x-y,\ 10xy,\ e^{x+100})$$

$$\therefore\ \vec{V}\cdot\hat{n} = \vec{V}\cdot\hat{z} = (x-y,\ 10xy,\ e^{x+100})\cdot(0,0,1) = e^{100+x}$$

$$\therefore\ \Phi_{total} = \int_{rectangle} e^{100+x}\, da \qquad da = dx\,dy$$



$$= \int_{y=2}^{\pi}\int_{x=2}^{5} e^{100+x}\, dx\,dy$$

$$= \int_{y=2}^{\pi} e^{100+x}\Big|_{x=2}^{5}\, dy$$

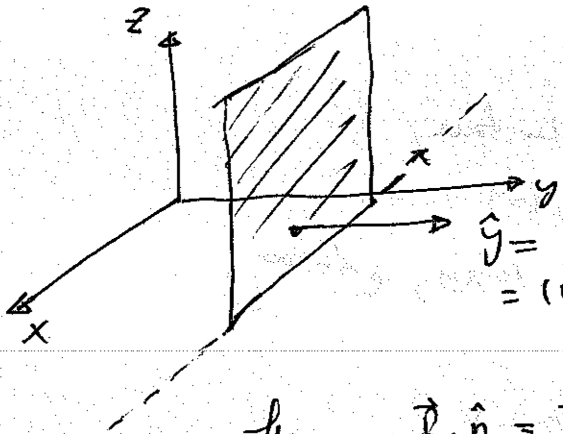
$$= \int_{y=2}^{\pi} [e^{105} - e^{98}]\, dy$$

$$= [e^{105} - e^{98}]\, y\Big|_{y=2}^{\pi}$$

$$= \boxed{(\pi - 2)(e^{105} - e^{98})}$$

(j) $\vec{R}(x,y,z) = (x - y + \sin(z),\ xz + xyz,\ x^3 + 2z + 3y)$

$\{(x,y,z) \mid 2 \le x \le 5,\ y = \pi,\ 0 \le z \le 5\}$



$\hat{y} = \hat{n} = (0,1,0)$

On rectangular face: $y = \pi$

so $\vec{R}(x, \pi, z) = (x - \pi + \sin(z),\ xz + \pi xz,\ x^3 + 2z + 3\pi)$.

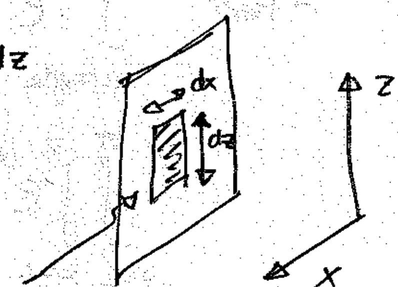
Hence, $\vec{R} \cdot \hat{n} = \vec{R} \cdot \hat{y}$

$= xz + \pi xz$

$= (1+\pi)xz$

$\therefore \Phi_{total} = \int_{\text{rectangular face}} (1+\pi)xz \, da$

$da = dx\,dz$



da = area of little pixel $= dx\,dz$

$= \int_{z=0}^{5}\int_{x=2}^{5} (1+\pi)xz \, dx\,dz$

$= (1+\pi) \left(\frac{x^2}{2}\right)\Big|_2^5 \left(\frac{z^2}{2}\right)\Big|_2^5$

$= (1+\pi)\left[\frac{25}{2} - \frac{4}{2}\right]^2$

$= \boxed{(1+\pi)\left(\frac{21}{2}\right)^2}$

$\boxed{19)}$ $\boxed{(a)}$ see notes.

$\boxed{(b)}$ See notes.

$\boxed{(c)}$ see notes.

$\boxed{(d)}$

$$K(x) = \overbrace{\underbrace{34}_{b_2=34}\sin(\underbrace{2}_{n=2}x) + \underbrace{100}_{b_{100}=100}\sin(\underbrace{100}_{n=100}x) + \underbrace{45}_{a_{23}=45}\cos(\underbrace{23}_{n=23}x) + \underbrace{100}_{b_{131}=100}\sin(\underbrace{131}_{n=131}x) + \underbrace{\cos}_{a_2=1}(\underbrace{2}_{n=2}x)}^{\text{This itself is a fourier series.}}$$

(All other a_n's and b_n's are zeros)

$$k(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}.$$

$\boxed{(e)}$

$$L(x) = \cos(2x)\cos(4x) - \sin(2x)\sin(4x) + \cos(100x)\sin(x) + \sin(100x)\cos(x)$$

$$= \cos(2x + 4x) + \sin(100x + x) \quad \leftarrow \text{using trig. identities}$$

$$= \underbrace{\underbrace{\cos}_{a_6=1}(\underbrace{6}_{n=6}x) + \underbrace{\sin}_{b_{101}=1}(\underbrace{101}_{n=101}x)}_{\text{fourier series itself!}}$$

(all other a_n's b_n's are zero).

> <u>Note</u>: The 2 trig. identities used above are:
>
> $$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
>
> $$\sin(a \pm b) = \cos(a)\sin(b) \pm \sin(a)\cos(b)$$