MITES 09 : Solution set 1

1.) $f(x) = \frac{1}{1+x}$

(i). $f(f(x)) = \frac{1}{1+f(x)} = \frac{1}{1+\frac{1}{1+x}} = \boxed{\frac{1+x}{2+x}}$ ← Makes sense for all but $x=-2$. ($\because f(x \to -2) = \pm\infty$).

(ii) $f\left(\frac{1}{x}\right) = \frac{1}{1+1/x} = \boxed{\frac{x}{1+x}}$

(iii) $f(cx) = \boxed{\frac{1}{1+cx}}$ (iv) $f(x+y) = \boxed{\frac{1}{1+x+y}}$ (v) $\boxed{\frac{1}{1+x} + \frac{1}{1+y} = f(x) + f(y)}$

(vi) $f(cx) = f(x)$

- For what "c" is there at least a single number x such that above equality ~~that~~ holds? ← Bit of a trick question.

$\frac{1}{1+cx} = \frac{1}{1+x} \Rightarrow cx = x$; Dividing both sides by x gives $c=1$

But there's more! since ~~above~~ $cx=x$ is true when $x=0$, for any value of c.

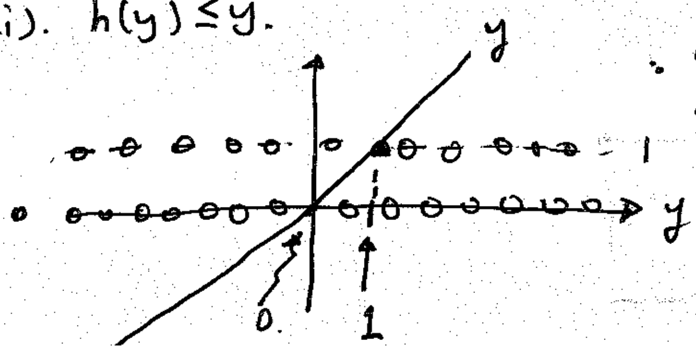
(we could not detect this case ~~[illegible]~~ by dividing both sides by x since you cannot divide by zero.)

$\therefore$ **Answer**: For any value of C, there is at least one number x that yields $f(cx) = f(x)$. [Namely, $x=0$].

(vii) From above, $\boxed{c=1}$ ~~[illegible]~~.

2.) $g(x) = x^2$; $h(x) = \begin{cases} 0 & , x \text{ rational} \\ 1 & , x \text{ irrational}. \end{cases}$

(i). $h(y) \le y$.



- Graphically, can see that we need $y \ge 0$.
- Also, for $y \ge 1$, $y \ge h(y)$ since $h(y) \le 1$.

· ~~Then~~ For $0 \leq y \leq 1$, need to look at when y is rational, and when y is irrational. [Note: $h(1) = 0$; $h(0) = 0$]

① • For $0 \leq y \leq 1$; and y ~~rational~~ irrational:

$$h(y) = 1. \Rightarrow h(y) > y.$$

② For $0 < y < 1$ and y irrational: $h(y) = 0 < y$.

∴ Summarizing our findings, we have:

$$\boxed{h(y) \leq y \text{ when } (y \geq 1) \text{ or } \left(\begin{array}{l} 0 \leq y \leq 1 \\ \text{and } y \text{ is rational.} \end{array}\right)}$$

(ii) $h(y) \leq g(y)$: $\Rightarrow h(y) \leq y^2$.

• Again, for $y \geq 1$, $y^2 \geq h(y)$ since $h(y) \leq 1$.

• For $0 \leq y < 1$: y rational: $h(y) = 0 \leq y^2$

y irrational: $h(y) = 1 > y^2$.

∴ $\boxed{h(y) \leq g(y) \text{ when } (y \geq 1) \text{ or } (0 \leq y \leq 1, \text{ and } y \text{ is rational})}$

(iii) $g(h(z)) = [h(z)]^2 = \begin{cases} 0, & \text{if } h(z) = 0 \\ 1, & \text{if } h(z) = 1 \end{cases}$ but $h(z) = \begin{cases} 0, & z \text{ rational} \\ 1, & z \text{ irrational} \end{cases}$

$$= \begin{cases} 0, & z \text{ rational} \\ 1, & z \text{ irrational} \end{cases}$$

$$= h(z)$$

∴ $\boxed{\begin{aligned} g(h(z)) - h(z) &= h(z) - h(z) \\ &= 0. \end{aligned}}$

(iv) $g(w) \le w$:

$\Rightarrow \quad w^2 \le w$

$\Rightarrow \quad w \le 1$ (if $w \neq 0$, and $w > 0$ divide both sides by w)

[Notice that if $w < 0$, then dividing both sides of $w^2 \le w$ gives $w \ge 1$.]

flipped inequality.

Notice that $w = 0$ also satisfies $w^2 \le w$.

So: $\boxed{g(w) \le w \text{ for } 0 \le w \le 1.}$

(v) $g(g(\varepsilon)) = g(\varepsilon)$:

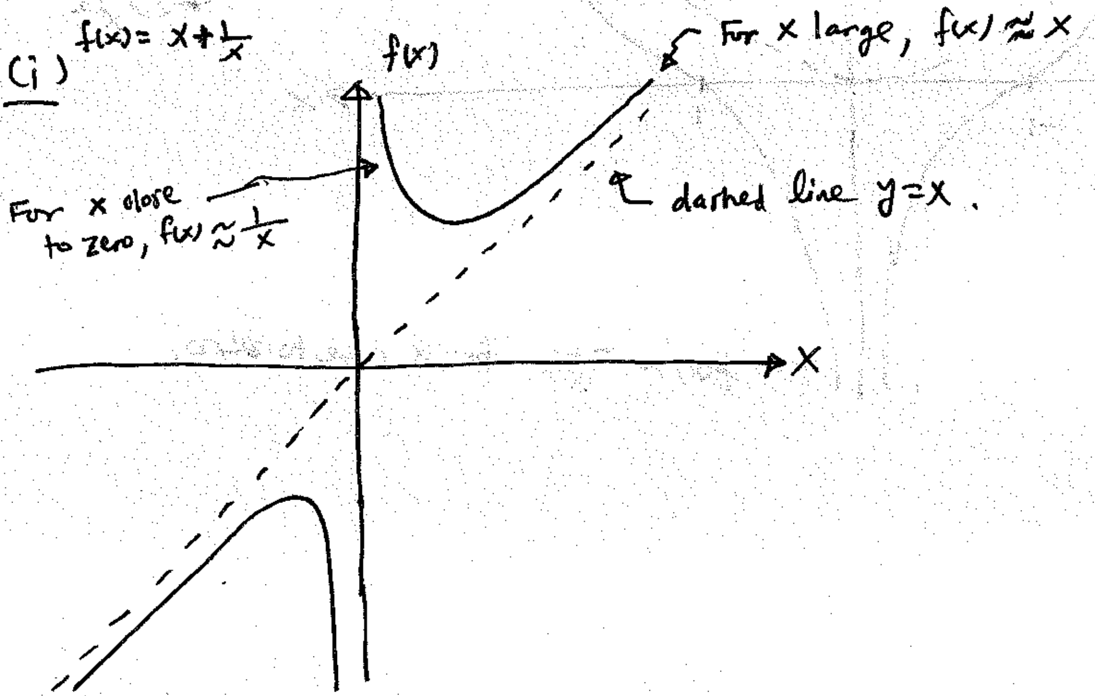
$\Rightarrow [g(\varepsilon)]^2 = \varepsilon^2 \quad \Rightarrow \quad \varepsilon^4 = \varepsilon^2$

$\Rightarrow \varepsilon = 0$ or $\varepsilon^2 = 1$

$\Rightarrow \varepsilon = \pm 1$

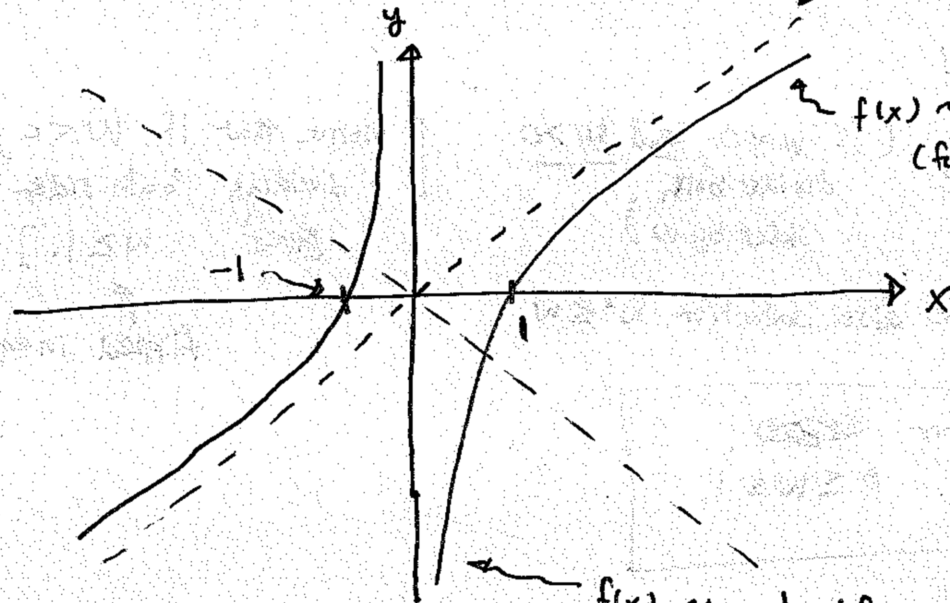
$\therefore \boxed{g(g(\varepsilon)) = g(\varepsilon) \text{ when } \varepsilon = 0, \text{ or } \varepsilon = \pm 1}$

3.) (i) $f(x) = x + \frac{1}{x}$



(ii) $f(x) = x - \frac{1}{x}$

dashed line $y = x$



$f(x) \sim x$ (below dashed line due to $-\frac{1}{x}$.)
(for x large)

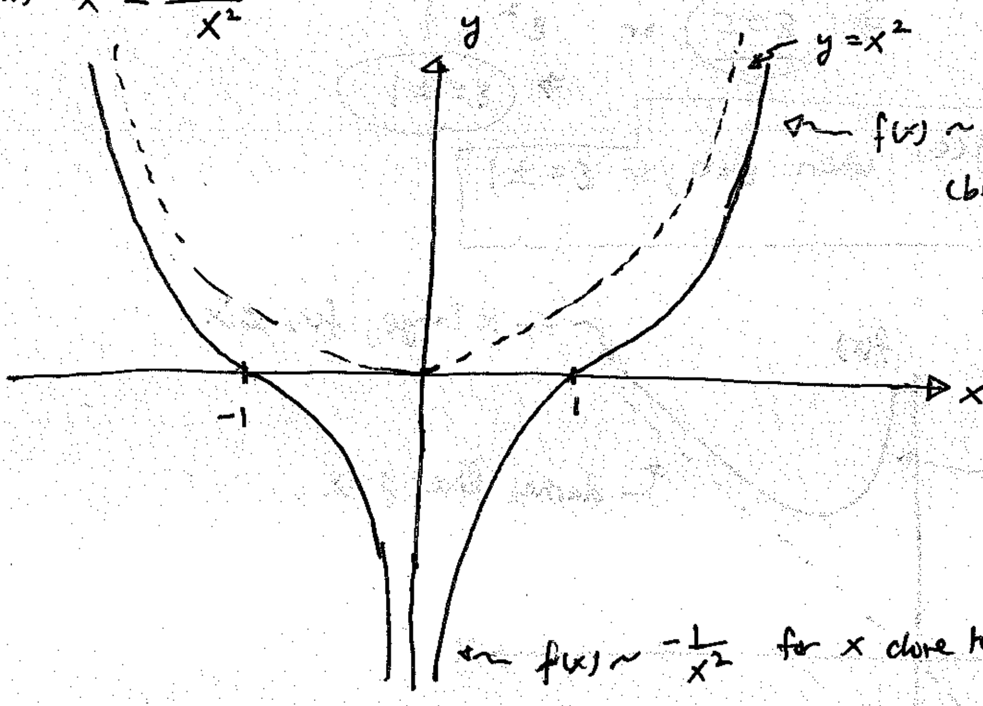
$f(x) \sim -\frac{1}{x}$ (for x near 0)

[Notice that $f'(x) = 0 \Rightarrow 1 = -1/x^2$ ← Not possible.
so no local max/min.]

(iii)

$f(x) = x^2 - \frac{1}{x^2}$

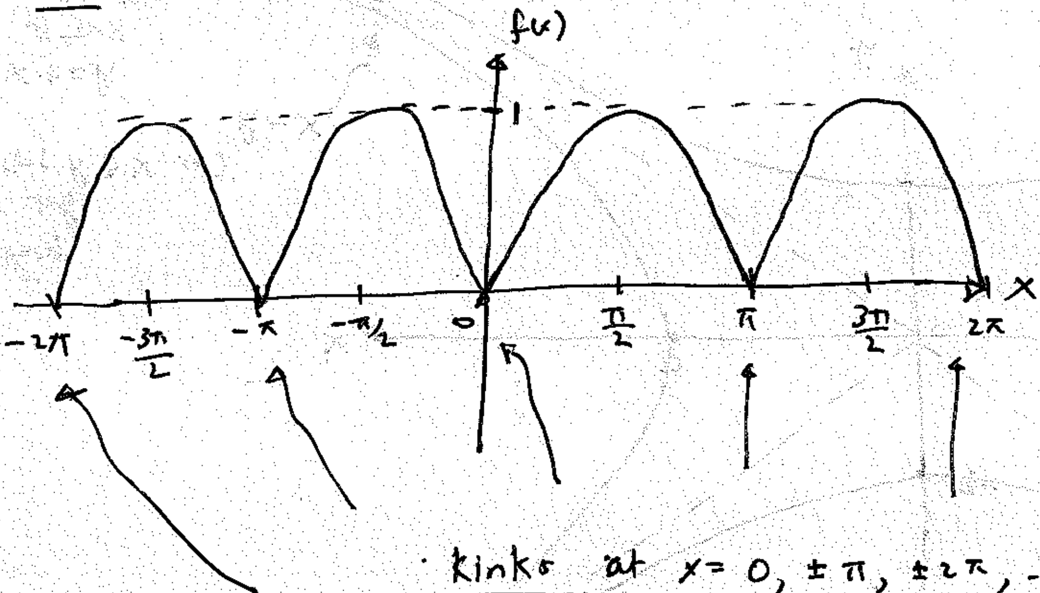
$y = x^2$



$f(x) \sim x^2$ for x large.
(below dashed ~~line~~ curve
due to $(-)\frac{1}{x^2}$

$f(x) \sim -\frac{1}{x^2}$ for x close to zero.

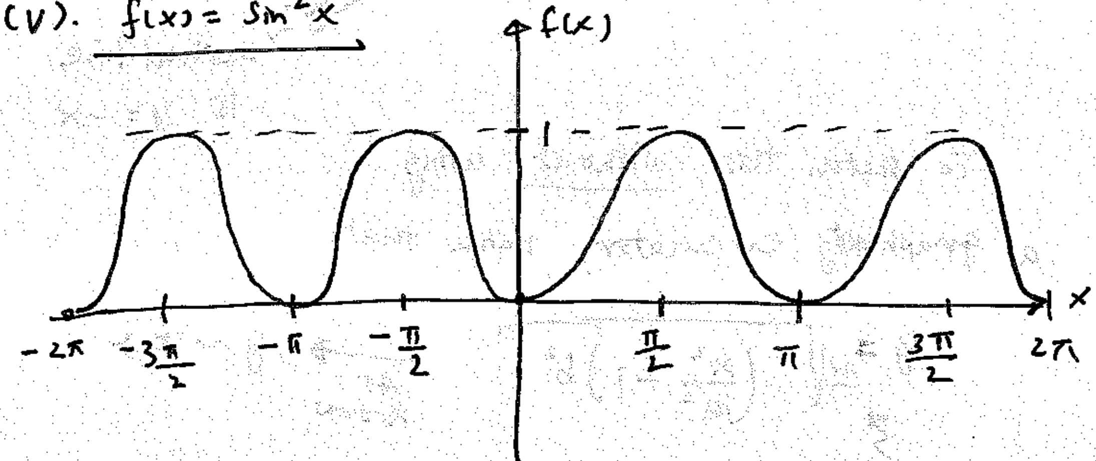
(iv) $f(x) = |\sin x|$



kinks at $x = 0, \pm\pi, \pm 2\pi, \ldots,$

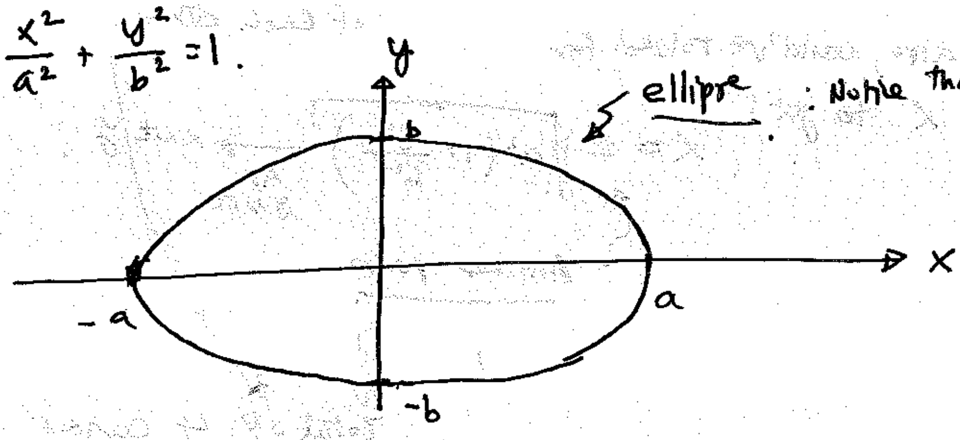
(i.e., $f(x)$ is not differentiable at these kinks.)

(v). $f(x) = \sin^2 x$



No more kinks. $f(x)$ is smooth everywhere.

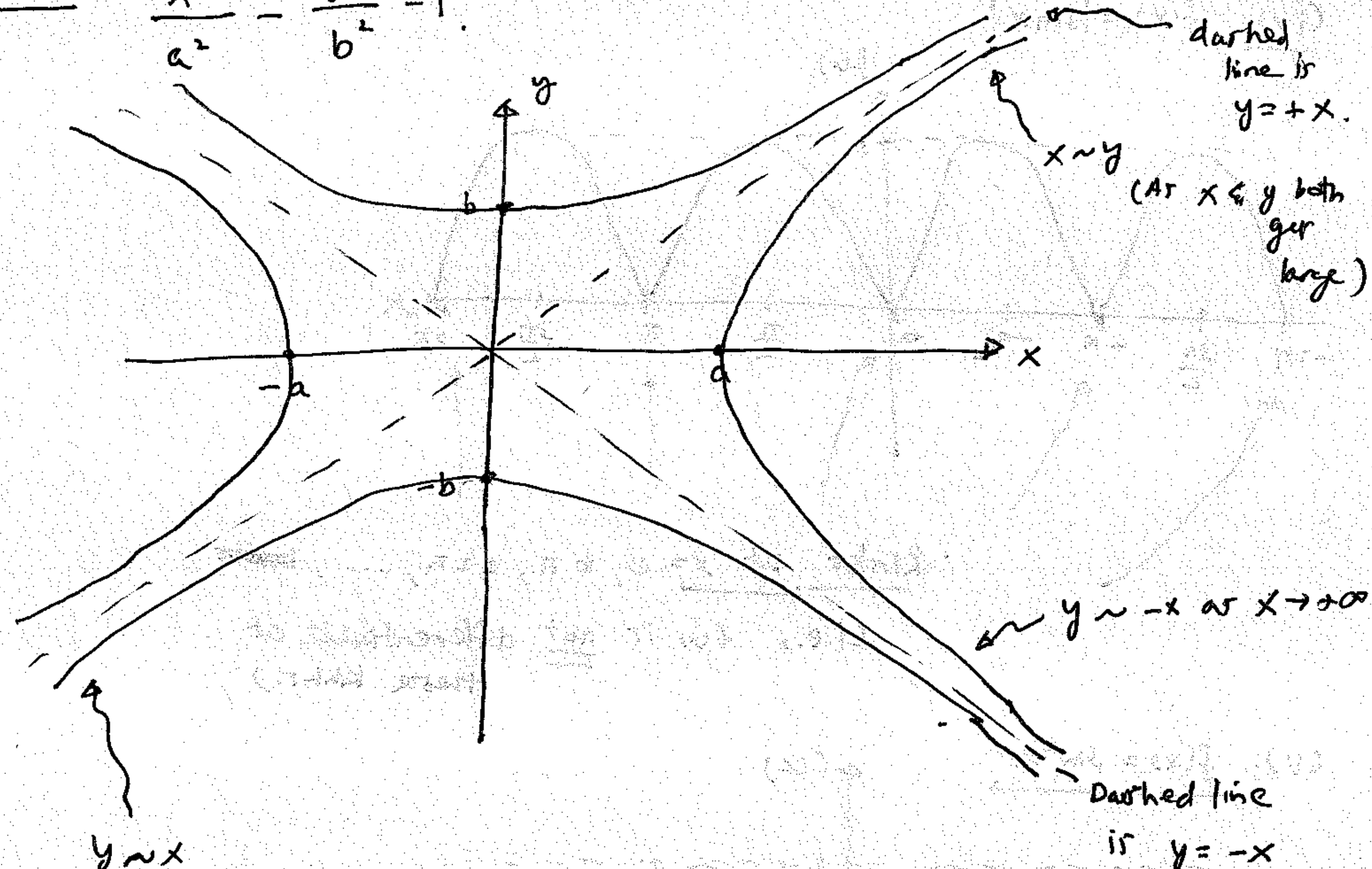
(vi). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



ellipse. Notice that when $a = b$, we get $x^2 + y^2 = r^2$.

circle.

(vii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$



To sketch this **without** using a graphing calculator, notice that

$$y = \pm\sqrt{\left(\frac{x^2}{a^2}-1\right)b^2} \xrightarrow[\text{As } x\to\infty]{} y \sim \pm x$$

2 possible signs, so we get a **pair** of curves that are **mirror images** of each other.

And also, could've solved for x to get:

$$x = \pm\sqrt{a^2\left(1+\frac{y^2}{b^2}\right)} \xrightarrow[\text{As } y\to\infty]{} x \sim \pm y.$$

Another pair

Total of 4 curves

4.) (i) $\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1-1}{2} = \boxed{0.}$

(ii) $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(ax^2+bx+c)}{\cancel{(x-2)}}$

To figure out a, b, c : Expand numerator to get: $x^3-8 = ax^3+(b-2a)x^2 - 2c$.

$\Rightarrow a=1$ $\quad b-2a=0$

$\Rightarrow b=2$ $\quad c=4$

$= \lim_{x\to 2} (x^2+2x+4) = 4+4+4 = \boxed{12}$.

(iii) $\lim_{x\to y} \frac{x^n-y^n}{x-y}$ (n some ⊕ integer).

Although it's ok to just apply the l'Hôpital's rule here, it's a much better testament to your calculus knowledge if you recognize that this is just the definition of derivative of $f(x)=x^n$, evaluated at $x=y$

i.e., $\lim_{x\to y} \frac{x^n-y^n}{x-y} = \lim_{h\to 0} \frac{(y+h)^n - y^n}{h} = \left.\frac{df}{dx}\right|_{x=y}$ where $f(x)=x^n$

by definition of derivative

$= \boxed{ny^{n-1}}$

(iv) $\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} = \left.\frac{df}{dx}\right|_{x=a}$ where $f(x)=\sqrt{x}$

Def^n of derivative

$= \left.\frac{1}{2x^{1/2}}\right|_{x=a} = \boxed{\frac{1}{2\sqrt{a}}}$

5.) We'll use
$$\alpha = \lim_{x\to 0} \frac{\sin x}{x}$$

(i) $$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)\cdot 2$$
$$= \lim_{(2x)\to 0} \frac{\sin(2x)}{(2x)}\cdot 2 = \boxed{2\alpha}$$

(ii) $$\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x\to 0} \left(\frac{bx}{\sin(bx)} \cdot \frac{\sin(ax)}{ax}\right)\cdot\left(\frac{ax}{bx}\right)$$
$$= \left[\lim_{x\to 0} \frac{bx}{\sin(bx)}\right]\cdot\left[\lim_{x\to 0} \frac{\sin(ax)}{ax}\right]\cdot \frac{a}{b}$$
$$= \underbrace{\left[\lim_{bx\to 0} \frac{(bx)}{\sin(bx)}\right]}_{=1/\alpha}\cdot\underbrace{\left[\lim_{ax\to 0} \frac{\sin(ax)}{ax}\right]}_{=\alpha}\cdot \frac{a}{b}$$
$$= \boxed{a/b}$$

(iii) $$\lim_{x\to 0} \frac{\sin^2 2x}{x} = \lim_{x\to 0} \left[\frac{\sin 2x}{2x}\cdot\frac{\sin 2x}{2x}\right]\underbrace{(4x)}$$
"4x"
$$= \underbrace{\left[\lim_{2x\to 0} \frac{\sin 2x}{2x}\right]^2}_{=\alpha^2}\cdot\underbrace{\left(\lim_{x\to 0} 4x\right)}_{=0}$$
$$= \boxed{0}$$

(iv) $\lim\limits_{x \to 0} \dfrac{\sin^2 2x}{x^2}$

$$= \lim_{x\to 0} \left[\frac{\sin 2x}{2x} \cdot \frac{\sin 2x}{2x} \right] \cdot 4 = \underbrace{\left[\lim_{2x \to 0} \frac{\sin(2x)}{(2x)} \right]^2}_{= \alpha^2} \cdot 4 = \boxed{4\alpha^2}$$

(v) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$; use $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\left[1 - \cos^2\left(\frac{x}{2}\right)\right] + \sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2(x/2)}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{\sin(x/2)}{(x/2)} \right]^2 \cdot 2 \cdot \frac{1}{4}$$

$$= \underbrace{\left[\lim_{\left(\frac{x}{2}\right) \to 0} \frac{\sin(x/2)}{(x/2)} \right]^2}_{= \alpha^2} \cdot \frac{1}{2} = \boxed{\frac{\alpha^2}{2}}$$

(vi) $\lim\limits_{x \to 0} \dfrac{\tan^2 x + 2x}{x + x^2}$

$$= \lim_{x \to 0} \frac{\cancel{x}\left[\frac{\sin^2 x}{x\cos^2 x} + 2\right]}{\cancel{x}\left[1 + x\right]}$$

$$= \lim_{x \to 0} \left\{ \frac{\left(\frac{\sin^2 x}{x^2}\right)\left(\frac{x}{\cos^2 x}\right) + 2}{1 + x} \right\} = \underbrace{\lim_{x \to 0} \frac{2}{1+x}}_{= 2} + \underbrace{\left[\lim_{x \to 0} \frac{\sin x}{x}\right]^2}_{= \alpha^2} \left[\lim_{x \to 0} \underbrace{\frac{x}{(1+x)}}_{\downarrow \frac{0}{1}} \underbrace{\frac{1}{\cos^2 x}}_{\downarrow 1} \right]$$

$$= 2 + 0 \cdot \alpha^2$$

$$= \boxed{2}$$

(vii) $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

From previous pg:

$1-\cos x = 2\sin^2(x/2)$

$= \lim_{x\to 0} \frac{x\sin x}{2\sin^2(x/2)}$

$= \lim_{x\to 0} \frac{(x/2)}{(\sin^2(x/2))}\cdot(\sin x)$

$= \lim_{x\to 0} \left[\frac{(x/2)}{\sin(x/2)}\right]^2 \cdot \frac{1}{(x/2)} \cdot \frac{(\sin x)}{x}\cdot x$

$= \underbrace{\left[\lim_{\frac{x}{2}\to 0}\frac{(x/2)}{\sin(x/2)}\right]^2}_{=\frac{1}{\alpha^2}} \cdot \underbrace{\left[\lim_{x\to 0}\frac{\sin x}{x}\right]}_{=\alpha} \cdot 2 = \boxed{\frac{2}{\alpha}}$

(viii) $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \frac{d(\sin x)}{dx} = \cos x$

Just the def^n of derivative.

But we want to ~~solve this~~ evaluate this limit differently; using "α":

To do: $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

$\sin(x+h) = \sin(x)\cos h + \cos x \sin h$

$= \lim_{h\to 0}\frac{(\sin x)[\cos h - 1]}{h} + \underbrace{\lim_{h\to 0}\frac{\cos x\sin h}{h}}_{=(\cos x)\cdot\alpha}$

$= \lim_{h\to 0}\frac{(\sin x)[2\sin^2(h/2)]}{h} + \alpha\cos x$ ← used $1-\cos h = 2\sin^2(h/2)$ (from previous problem.)

$= (\sin x)\underbrace{\left(\lim_{h\to 0}\frac{h}{2}\right)}_{\to=0}\cdot\underbrace{\left(\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}\right)^2}_{=\alpha^2} + \alpha\cos x$

So: $= \overbrace{0\cdot(\sin x)\cdot\alpha^2}^{\to 0} + \alpha\cos x$

$= \alpha \cos x$

$\therefore \lim\limits_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h} = \boxed{\alpha \cos x}$ ← Ans. in terms of α.

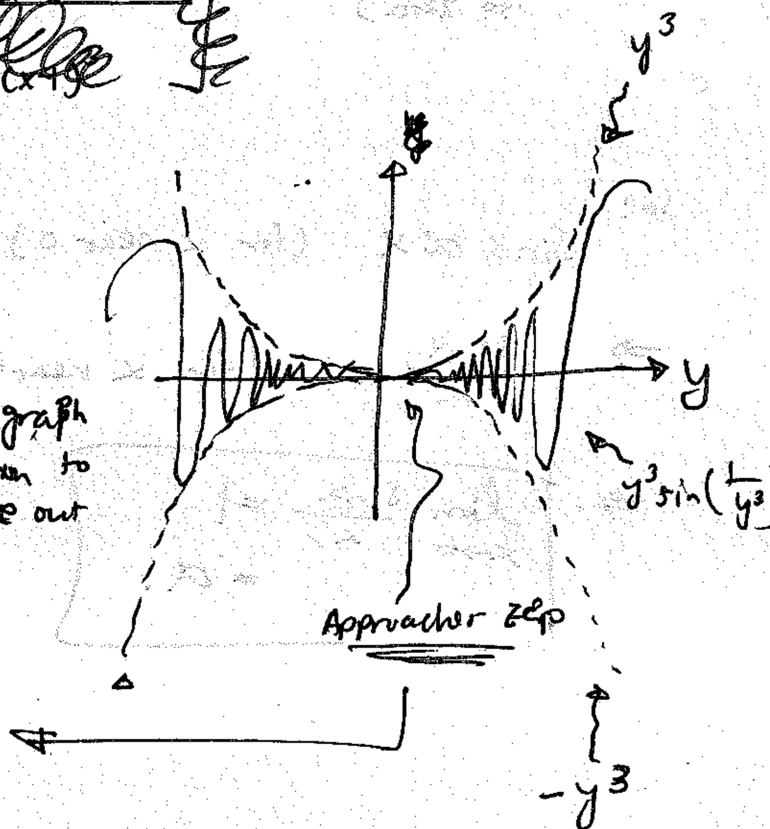
(ix) $\lim\limits_{x\to 1} (x^2-1)^3 \sin\left(\dfrac{1}{(x-1)^3}\right)$

$= \lim\limits_{x\to 1} (x-1)^3(x+1)^3 \sin\left(\dfrac{1}{(x-1)^3}\right) = \left[\lim\limits_{x\to 1} (x+1)^3\right]\cdot\left[\lim\limits_{x\to 1} (x-1)^3 \sin\left(\dfrac{1}{(x-1)^3}\right)\right]$

$= 8 \cdot \lim\limits_{(x-1)\to 0} (x-1)^3 \sin\left(\dfrac{1}{(x-1)^3}\right)$

$= 8 \lim\limits_{y\to 0} \underbrace{y^3 \sin\left(\dfrac{1}{y^3}\right)}$.

↑ look at the graph of this function to figure out



$= 8\cdot 0$

$= \boxed{0.}$

← Ans.

Taylor series of $\sin x$ can tell us why $\alpha = 1$:

$\sin x = f(x)$

so:

$$f(x) = f^{(0)}(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)\,x^2}{2!} + \frac{f^{(3)}(0)\,x^3}{3!} + \cdots$$

$$= \underbrace{\cancel{\sin(0)}}_{0} + \underbrace{\cos(0)}_{=1} x + \frac{\underbrace{\cancel{(-\sin(0))}}_{0} x^2}{2!} - \frac{\cos(0)\,x^3}{3!} + \cdots$$

$$= x - \frac{x^3}{3!} + \text{higher order terms of form } \# x^n \quad \underline{n > 3}.$$

$\approx x$

(when x close to zero.)

All these terms are much much smaller than x when $x \to 0$

To see this:

note that $\lim_{x \to 0} \frac{x^n}{x} = x^{n-1}$

$= 0$

(if $n > 1$.)

So:

$\sin x \approx x$ (for x near 0)

$\Rightarrow \frac{\sin x}{x} \approx 1$ (for x near 0).

$$\Rightarrow \boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1 = \alpha.}$$

$\therefore \boxed{\alpha = 1}$

6.)

(i) $f(x) = x^4$; $l = a^4$.

Given $\varepsilon > 0$, need to find $\delta > 0$ such that:

$|f(x) - l| < \varepsilon$ as long as $0 < |x - a| < \delta$.

• First, start with $|f(x) - l| = |x^4 - a^4|$

We have $\varepsilon > |x^4 - a^4|$

$= |x^2 - a^2| \cdot |x^2 + a^2|$ since $|x^2 + a^2| \geq a^2$

$\geq a^2 |x^2 - a^2|$

$= a^2 |x - a| \cdot |x + a|$

$\geq a^2 |x - a|^2$

Now, as long as we pick δ so that $a - \delta > 0$ (if $a > 0$) or $a + \delta < 0$ (if $a < 0$) ~~that~~ we have $|x + a| \geq |x - a|$ ~~that~~ so

Note that if $a = 0$, this is also true since $|x + a| \geq |x - a|$ still true (i.e., $|x + 0| = |x - 0|$)

so: $\varepsilon > a^2 |x - a|^2$

$\Rightarrow \frac{\varepsilon}{a^2} > |x - a|^2$

$\Rightarrow \left(\frac{\varepsilon}{a^2}\right)^{1/2} > |x - a| > 0$.

So, pick δ such that it's equal to or lgr than $\left(\frac{\varepsilon}{a^2}\right)^{1/2}$

and also need $a - \delta > 0$ (if $a > 0$) (see above)

or $a + \delta < 0$ (if $a < 0$)

We can do this by noticing that if $0 < \delta < \frac{|a|}{2}$, then certainly $a - \delta > 0$ (if $a > 0$)

(when $a \neq 0$) $a + \delta < 0$ (if $a < 0$)

So: pick $\boxed{\delta = \min\left(\frac{|a|}{2}, \left(\frac{\varepsilon}{a^2}\right)^{1/2}\right)}$

means pick the smaller of these 2 numbers.

(ii) $f(x) = \frac{1}{x}$; $a=1$, $l=1$.

Given $\varepsilon > 0$:

$$\varepsilon > \left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| = \frac{|x-1|}{|x|} > \frac{|x-1|}{100}$$

Notice that we can pick δ to be as small as we want, as long as $\varepsilon > \left|\frac{1}{x} - 1\right|$ is satisfied.

(we'll want $0 < |x-1| < \delta$.)

we'll pick δ to be small enough so that $|x| < 100$.

$\Rightarrow \frac{1}{|x|} > \frac{1}{100}$.

so

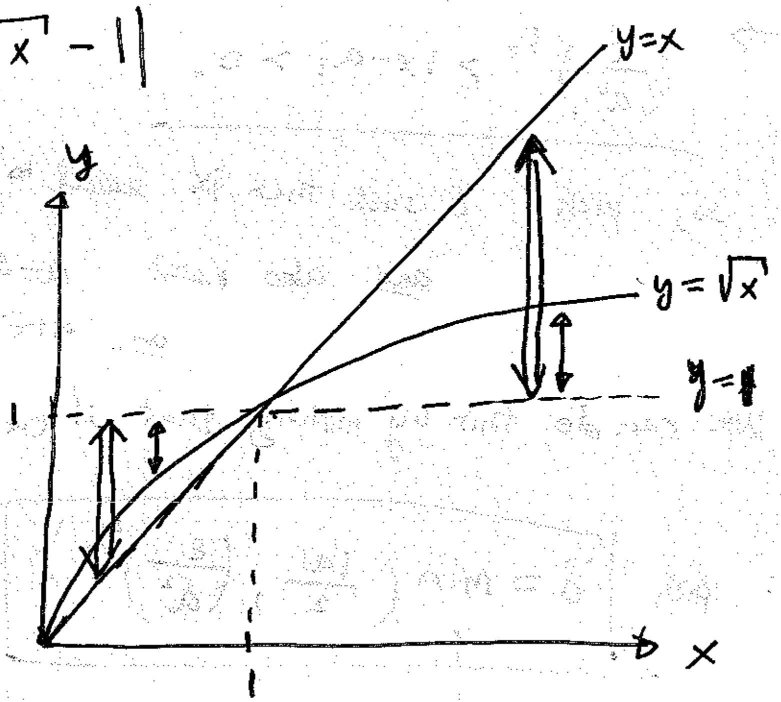
$\Rightarrow$ $100\varepsilon > |x-1| > 0$.

so, pick $\boxed{\delta = \min(100\varepsilon, 99)}$

(iii) $f(x) = \sqrt{x}$; $a=1$, $l=1$.

Given $\varepsilon > 0$: $\varepsilon > |\sqrt{x} - 1|$

Consider the graph:



Notice that the distance between $y=x$ and $y=1$ line is always larger than the distance between $\sqrt{x} = y$ curve and $y=1$ line.

i.e., $|\sqrt{x} - 1| \leq |x-1|$

equal only when $x=1$.

so: $\left|\sqrt{x}-1\right| < |x-1| < \delta$

i.e. For all x such that $0<|x-1|<\delta$, we have

$\left|\sqrt{x}-1\right| < |x-1| < \delta$.

(we also have to make $\delta < 1$ for otherwise, we might get $x<0$, but we can't allow this since $\sqrt{x}$ is undefined for $x<0$)

(If we pick $\delta=\varepsilon$, then $\left|\sqrt{x}-1\right| < |x-1| < \varepsilon$.

So, pick $\boxed{\delta = \min(\varepsilon, 1)}$

7.) Details of why l'Hôpital rule is true is covered in Lecture note #1 (involves approximation of function by its tangent line).

8.) (i) $f(x)=\sin(x+x^2)$

$$\boxed{\frac{df}{dx} = \left[\cos(x+x^2)\right]\cdot\left[1+2x\right]}$$

(ii) $f(x) = \sin\left(\frac{\cos x}{x}\right)$

$$f'(x) = \cos\left[\frac{\cos x}{x}\right]\cdot\frac{d}{dx}\left(\frac{\cos x}{x}\right)$$

$$= \left[\cos\left(\frac{\cos x}{x}\right)\right]\cdot\left\{\frac{-x\sin x - \cos x}{x^2}\right\}$$

$$\boxed{f'(x) = \left[\cos\left(\frac{\cos x}{x}\right)\right]\cdot\left[\frac{-x\sin x - \cos x}{x^2}\right]}$$

(iii) $f(x) = \sin^3(\sin^2(\sin x))$

$f(u) = \sin^3(u)$. ; $u = \sin^2(\sin x)$

So: $\frac{df}{dx}(x) = \frac{df(u)}{du}\cdot\frac{du}{dx} = 3\sin^2(u)\cos(u)\cdot\frac{du}{dx}$

over

$$u = \sin^2(\sin x) = \sin^2(v)$$

$$\frac{dv}{dx} = \cos x$$

$$\frac{df}{dx} = 3\sin^2(u)\cos(u)\,\frac{du}{dx}$$

$$= 3\sin^2(u)\cos(u)\,\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= \boxed{[3\sin^2(u)\cos(u)]\,[2\sin(v)\cos(v)]\cos x}$$

(iv) $f(x) = \frac{1}{1+x}$.

$$\boxed{\frac{df}{dx} = \frac{-1}{(1+x)^2}}$$

9.) (i). $f(x) = g(x + g(a))$ $\quad$ a = constant.

$$f' = \frac{df}{dx}$$

$$= \frac{d}{dx}[g(x + g(a))]$$

$$= \frac{dg(u)}{du}\cdot\underbrace{\frac{du}{dx}}_{=1} \quad ; \quad u = x + g(a)$$

$$\frac{du}{dx} = \frac{dx}{dx} = 1.$$

$$= \boxed{g'(x+g(a))}$$

$g' = \frac{dg(u)}{du}$ ← same as $\frac{dg(x)}{dx}$ ← Important that you know why this is true.

(ii) $f(x) = g(x\cdot g(a))$

$$f' = \frac{df}{dx} = \frac{d}{dx}[g(\underbrace{x\cdot g(a)}_{u})]$$

$$= \frac{dg}{du}\cdot\frac{du}{dx}$$

$$\frac{du}{dx} = \frac{d}{dx}(g(a)\cdot x) = g(a)$$

$$= \boxed{g(a)\,g'(x\,g(a))}$$

(iii) $f(x) = g(x + g(x))$

$$f'(x) = \frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} \qquad u = x + g(x) \qquad \frac{du}{dx} = 1 + g'(x)$$

$$= \boxed{[1 + g'(x)] \cdot g'(x + g(x))}$$

(iv) $f(x) = g(x)(x-a)$

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}[g(x)(x-a)]$$

$$= g'(x)(x-a) + g(x)\underbrace{\frac{d}{dx}(x-a)}_{=1} \quad \leftarrow \text{chain rule}$$

$$= \boxed{g(x) + g'(x)(x-a)}$$

10.) (i) $f(x) = x^3 - x^2 - 8x + 1$. on $[-2, 2]$ x-interval.

$f(-2) = -8 - 4 + 16 + 1$
$= 5$

$f(2) = 8 - 4 - 16 + 1$
$= -11$

$$f'(x) = 3x^2 - 2x - 8$$

$0 = f'(x) \Rightarrow 0 = 3x^2 - 2x - 8$

solve for root using quadratic root finding eqn.

$$\Rightarrow x_{root} = \frac{2 \pm \sqrt{4 - 4(3)(-8)}}{6}$$

$$= \frac{2 \pm 2\sqrt{1 + 24}}{6}$$

$$= \frac{2 \pm 10}{6}$$

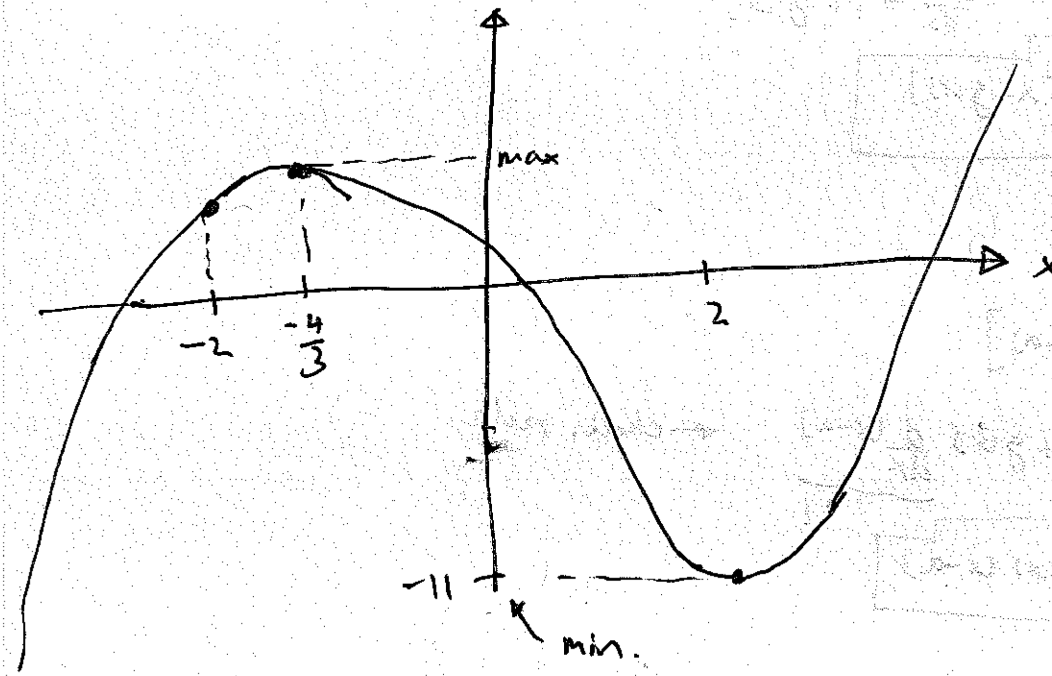
$x_- = -\frac{4}{3}$

$x_+ = 2$

both within $[-2, 2]$ interval

2 turning points.

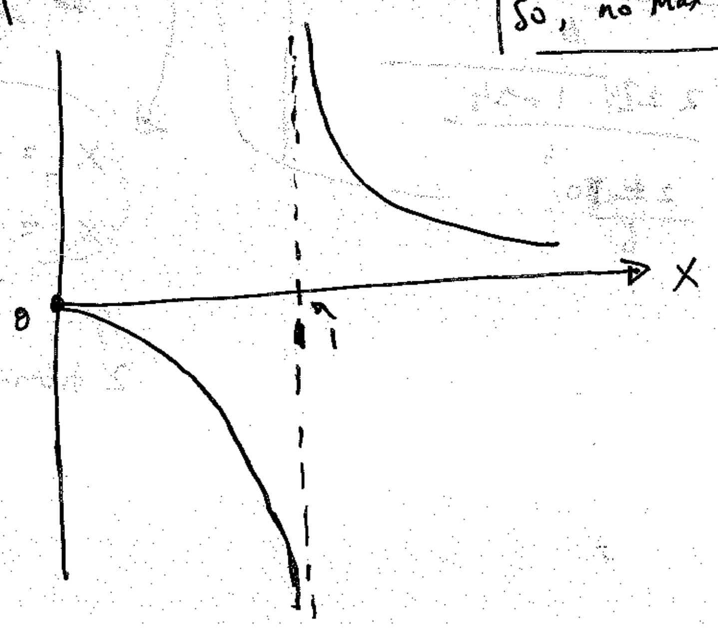
we now have enough info. to graph the function and get our answer:



- Max within [-2,2] : occurs at $x = -\frac{4}{3}$; $f(-\frac{4}{3}) = (-\frac{4}{3})^3 - (-\frac{4}{3})^2 + 8(\frac{4}{3}) + 1$
- Min within [-2, 2] : occurs at $x = 2$. $f(2) = -11$

(ii) $f(x) = \frac{x}{x^2-1}$ on [0, 5]

So, no max or min on [0, 5]



11.)

(i) $\displaystyle\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}}\,dx$

$= \displaystyle\int \left(x^{\frac{6-5}{10}} + x^{\frac{1-3}{6}}\right) dx$

$= \displaystyle\int \left[x^{1/10} + x^{-1/3}\right] dx$

$= \boxed{\frac{10}{11} x^{11/10} + \frac{3}{2} x^{2/3} + C.}$

constant.

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1} + \sqrt{x+1}}$

$= \displaystyle\int \frac{\sqrt{x-1} - \sqrt{x+1}}{(\sqrt{x-1} + \sqrt{x+1})} \frac{1}{(\sqrt{x-1} - \sqrt{x+1})} dx$

$= \displaystyle\int \frac{\sqrt{x-1} - \sqrt{x+1}}{(x-1) - (x+1)} dx$

$= \displaystyle\int dx \left[\frac{-\sqrt{x-1}}{2} + \frac{\sqrt{x+1}}{2}\right]$

$= \frac{1}{2}\left(\frac{2}{3}\left[-(x-1)^{3/2}\right] + \frac{2}{3}(x+1)^{3/2}\right)$

$= \boxed{\frac{1}{3}\left[-(x-1)^{3/2} + (x+1)^{3/2}\right] + C.}$

(iii) $\displaystyle\int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}}\,dx$

$= \displaystyle\int e^{-3x} + e^{-2x} + e^{-x}\,dx$

$= \boxed{\frac{e^{-3x}}{-3} - \frac{e^{-2x}}{2} - e^{-x} + C.}$

(iv) $\displaystyle\int \frac{dx}{a^2+x^2} = \int \frac{(1/a)^2\,dx}{1+(\frac{x}{a})^2}$

$= \frac{1}{a}\displaystyle\int \frac{d(x/a)}{1+(x/a)^2}$

$= \frac{1}{a}\displaystyle\int \frac{du}{1+u^2} \qquad u = x/a$

Now, note:

letting $u = \tan y$.

$\Rightarrow 1+u^2 = \sec^2 y = \frac{1}{\cos^2 y}$

and $\frac{du}{dy} = \frac{d}{dy}\left(\frac{\sin y}{\cos y}\right) = \sec^2 y = \frac{1}{\cos^2 y}.$

$\Rightarrow du = \frac{dy}{\cos^2 y}.$

$\therefore \frac{1}{a}\displaystyle\int \frac{du}{1+u^2}$

$= \frac{1}{a}\displaystyle\int \frac{\cancel{\cos^2 y}\,dy}{\cancel{\cos^2 y}}$

$= \frac{1}{a} y + C. = \frac{1}{a}\arctan(u) + c$

$= \boxed{\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$

12.)

(i) $\int e^x \sin(e^x)\,dx$ $\qquad e^x \equiv u$

$\frac{du}{dx} = e^x$

$\Rightarrow \frac{du}{u} = dx$

$= \int \frac{\cancel{u}\sin(u)}{\cancel{u}}\,du$

$= \cancel{-\cos(u) + C.} = \boxed{-\cos(e^x)+C}.$

(ii) $\int x e^{-x^2} dx$ $\qquad$ Let $u = x^2$

$\Rightarrow du = 2x\,dx$

$= \int \frac{du}{2} e^{-u}$

$= -\frac{1}{2} e^{-u} + C.$

$= \boxed{-\frac{e^{-x^2}}{2} + C.}$

(iii) $\int \frac{\log x}{x}\,dx$ $\qquad$ Let $u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$\Rightarrow du = \frac{dx}{x}$

$= \int du \cdot u$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{(\log x)^2}{2} + C.}$

(iv) $\int \frac{e^x dx}{e^{2x} + 2e^x + 1}$ $\qquad$ Let $u = e^x$

$du = u\,dx$

$\Rightarrow \frac{du}{u} = dx$

$= \int \frac{du}{u^2 + 2u + 1}$

$= \int \frac{du}{(u+1)^2} = -\frac{1}{(u+1)} + C$

$= \boxed{\frac{-1}{e^x+1} + C}$

(v) $\int e^{e^x} e^x\,dx$ $\qquad$ Let $e^x = u.$

$= \int e^u\,du$

$= e^u + C$

$= \boxed{e^{e^x} + C}$

(vi) $\int \log(\cos x)\tan x\,dx$ $\qquad$ Let $u = \cos x$

$\Rightarrow \frac{du}{dx} = -\sin x$

$\Rightarrow du = -\sin x\,dx$

$= \int \log(u)\frac{\sin x\,dx}{\cos x}$

$= \int \frac{-\log(u)}{u}\,du$ $\qquad$ from (iii)

$= -\frac{(\log u)^2}{2} + C$

$\Rightarrow = \boxed{-\frac{[\log(\cos x)]^2}{2} + C.}$

13.)

(i) $\int \underbrace{x^2}_{g} \underbrace{e^x}_{f'} dx$

$= e^x x^2 - \int 2x e^x dx$

$= e^x x^2 - \left[2x e^x - \int 2e^x dx \right]$

$= \boxed{e^x x^2 - 2x e^x + 2e^x + c.}$

constant.

(ii) $\int \underbrace{x^2}_{g} \underbrace{\sin x}_{f'} dx$

$= -x^2 \cos x + \int 2x \cos x dx$

$= -x^2 \cos x + \left[2x \sin x - \int 2 \sin x dx \right]$

$= \boxed{-x^2 \cos x + 2x \sin x + 2 \cos x} + c$

(iii) $\int (\log x)^3 dx$

$= \int \underbrace{1}_{\text{"}f'\text{"}} \cdot \underbrace{(\log x)^3}_{\text{"}g\text{"}} dx$

$= x(\log x)^3 - 3 \int \frac{\cancel{x} (\log x)^2}{\cancel{x}} dx$

$= x(\log x)^3 - 3 \left[x(\log x)^2 - 2 \int \frac{\cancel{x}}{\cancel{x}} \log x \, dx \right]$

$= x(\log x)^3 - 3 \left[x(\log x)^2 - 2 \left\{ x \log x - \underbrace{\int dx}_{\text{"}x} \right\} \right]$

$= \boxed{x(\log x)^3 - 3x(\log x)^2 + 6x \log x - 6x + c}$

(iv) $\int \cos(\log x) dx$

let $u = \log x$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow x \, du = dx$

but $e^u = x$

so: $e^u du = dx$

$= \int \underbrace{\cos(u)}_{\text{"}g} \underbrace{e^u}_{\text{"}f'} du$

$= e^u \cos u + \int e^u \sin u \, du$

$= e^u \cos u + e^u \sin u - \int e^u \cos u \, du$

so we have: $\int (\cos u) e^u du = e^u [\cos u + \sin u] - \int e^u \cos u \, du$

$\Rightarrow \quad 2 \int \cos u \, e^u du = e^u [\cos u + \sin u]$

$\Rightarrow \int \cos(\log x) dx = \frac{e^u}{2} [\cos u + \sin u]$

$= \boxed{\frac{x}{2} \left[\cos(\log x) + \sin(\log x) \right] + c}$

(v) $\int \underbrace{\sqrt{x}}_{\text{"f'"}} \underbrace{\log x}_{\text{"g"}} dx$

$= \frac{2}{3}x^{3/2}\log x - \int \frac{2}{3}\underbrace{\frac{x^{3/2}}{x}}_{= x^{1/2}} dx$

$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2}$

$= \boxed{\frac{2}{3}x^{3/2}\left[\log x - \frac{2}{3}\right]} + C.$

(vi) $\int \underbrace{x}_{\text{"f'"}} \underbrace{(\log x)^2}_{\text{"g"}} dx$

$= \frac{x^2}{2}(\log x)^2 - \int \frac{x^{\cancel{2}}}{\cancel{2}}\frac{\cancel{2}\log x}{\cancel{x}} dx$

$= \frac{x^2}{2}(\log x)^2 - \left\{\frac{x^2}{2}\log x - \int \frac{x}{2} dx\right\}$

$= \boxed{\frac{x^2}{2}\log x\left[\log x - 1\right] + \frac{x^2}{4} + C}$

14.) (i) $\int \frac{dx}{\sqrt{1-x^2}}$ let $x = \sin u$, $dx = \cos u \, du$

$= \int \frac{\cancel{\cos u}\, du}{\cancel{\cos u}}$

$= u + C = \boxed{\arcsin(x) + C}$

(ii) $\int \frac{dx}{\sqrt{1+x^2}}$ Let $x = \tan u$, $dx = \sec^2 u \, du$

$= \int \frac{\sec^{\cancel{2}} u \, du}{\cancel{\sec u}}$

$= \int \sec u \, du$

$= \log(\sec u + \tan u)$ ← given on the Problem set handout

$= \boxed{\log\left[\sec(\tan^{-1}x) + x\right]} + C$

(iii) $\int \frac{dx}{\sqrt{x^2-1}}$ let $x = \sec u$

$\frac{dx}{du} = \frac{d}{du}\left[\frac{1}{\cos u}\right]$

$= \frac{+1}{\cos^2 u}\sin u$

$\Rightarrow dx = \frac{\sin u}{\cos^2 u} du$

$= \int \frac{\sin u \, du}{\cos^2 u \tan u}$

$= \int \frac{\cancel{\sin u}}{\cos^{\cancel{2}} u}\frac{\cancel{\cos u}}{\cancel{\sin u}} du$

$= \int \sec u \, du$

$x = \frac{1}{\cos u} \Rightarrow u = \mathrm{ArcCos}\left(\frac{1}{x}\right)$

$= \log[\sec u + \tan u] + C$

$= \log\left[\sec\left(\mathrm{ArcCos}\left(\frac{1}{x}\right)\right) + \tan\left(\mathrm{ArcCos}\left(\frac{1}{x}\right)\right)\right] + C$

$= \boxed{\log\left[x + \tan\left(\mathrm{ArcCos}\left(\frac{1}{x}\right)\right)\right] + C}$

iv) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ Let $x = \sec u$ $\displaystyle\frac{dx}{du} = \frac{+\sin u}{\cos^2 u}$

$\displaystyle = \int \frac{\sin u \, du}{\cos^2 u \, \sec u \, \tan u}$

$\displaystyle = \int \frac{\cancel{\sin u} \quad \cancel{\cos u}}{\cancel{\cos^2 u}\left(\frac{1}{\cancel{\cos u}}\right)\cancel{\sin u}} \, du$

$= u + C.$

$= \boxed{\operatorname{Arcsec}(x) + c}$

(v) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$ let $x = \sin u$
$dx = \cos u \, du$

$\displaystyle = \int \sin^3 u \cos^2 u \, du$

$\displaystyle = \int \sin u \, [1-\cos^2 u] \cos^2 u \, du$

$\displaystyle = \int \sin u \, [\cos^2 u - \cos^4 u] \, du$

$\displaystyle = \int \cos^2 u \sin u \, du - \int \cos^4 u \sin u \, du$

$\displaystyle = \frac{-\cos^3 u}{3} - \left(\frac{-\cos^5 u}{5}\right)$

$\displaystyle = \boxed{\frac{\cos^5 u}{5} - \frac{\cos^3 u}{3}} + C.$

$\displaystyle = \boxed{\frac{\cos^5(\sin^{-1}(x))}{5} - \frac{\cos^3(\sin^{-1}(x))}{3} + C}$

(vi) let $x = \sin u$
$dx = \cos u \, du$

$\displaystyle\int \sqrt{1-x^2}\,dx$

$\displaystyle = \int \cos^2 u \, du$

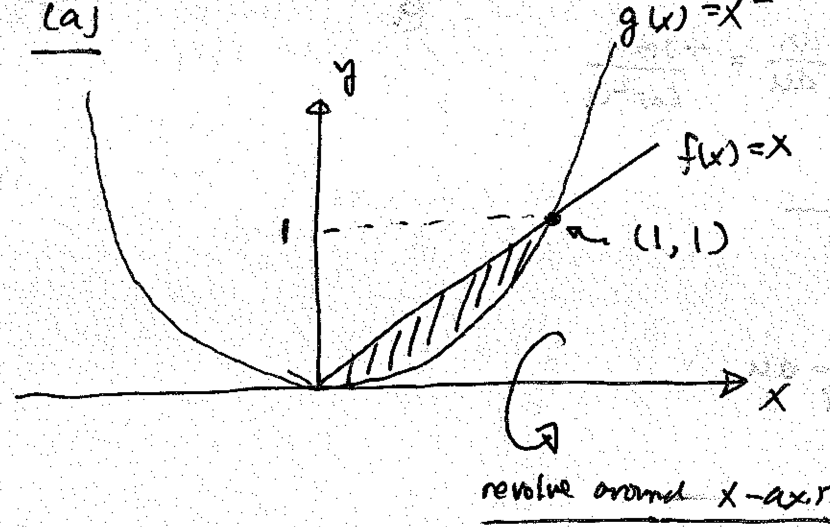
see lecture note #2

$\displaystyle = \int \left[\frac{1+\cos 2u}{2}\right] du$

$\displaystyle = \frac{u}{2} + \frac{\sin 2u}{4} + C$

$\displaystyle = \boxed{\frac{\operatorname{Arcsin}(x)}{2} + \frac{\sin(2\operatorname{Arcsin}(x))}{4} + C.}$

15.) (a)



Shaded region: ~~rotate about~~ ← region bounded by f & g.

$$\pi\,[g(x) - f(x)]^2\,dx = dV$$

is

$$\Rightarrow V = \sum dV$$

$$= \int_0^1 \pi\,[g(x) - f(x)]^2\,dx$$

$$= \int_0^1 \pi\,[x^2 - x]^2\,dx$$

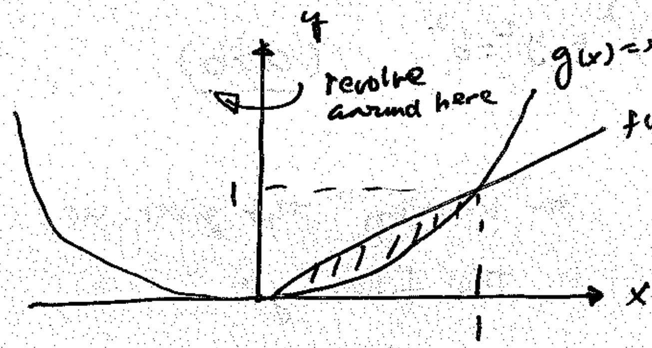
$$= \int_0^1 \pi\,[x^4 - 2x^3 + x^2]\,dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]\Bigg|_0^1$$

$$= \pi\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \pi\left[\frac{1}{30}\right] \longrightarrow \boxed{\text{Vol.} = \frac{\pi}{30}}$$

(b) This time, revolve around y-axis:



$g(x) = x^2 \rightarrow y = x^2 \Rightarrow \sqrt{y} = x$

$f(x) = x \rightarrow y = x$

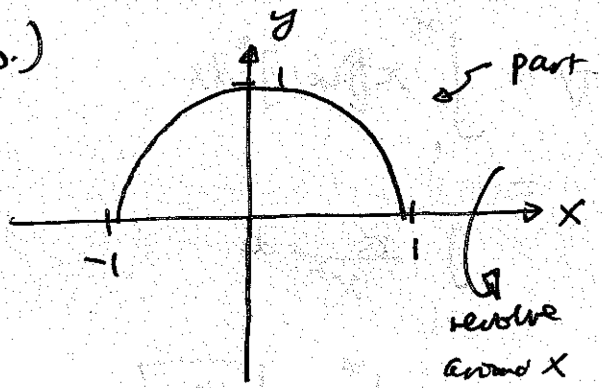
$$dV = \pi\left[\sqrt{y} - y\right]^2 dy$$

$$V = \int_0^1 dV = \int_0^1 \pi\left[\sqrt{y} - y\right]^2 dy = \int_0^1 \pi\left[y - 2\sqrt{y}\,y + y^2\right] dy$$

$$= \pi\left[\frac{y^2}{2} - 2y^{5/2}\left(\frac{2}{5}\right) + \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right] = \boxed{\frac{\pi}{30}}$$

16.)



part of $x^2 + y^2 = 1$

$\rightarrow y = \sqrt{1 - x^2}$

$\uparrow$ (+) root to describe the upper ($x>0$) part of unit circle

Vol. generated by revolve around x-axis is: $V = \int dV$

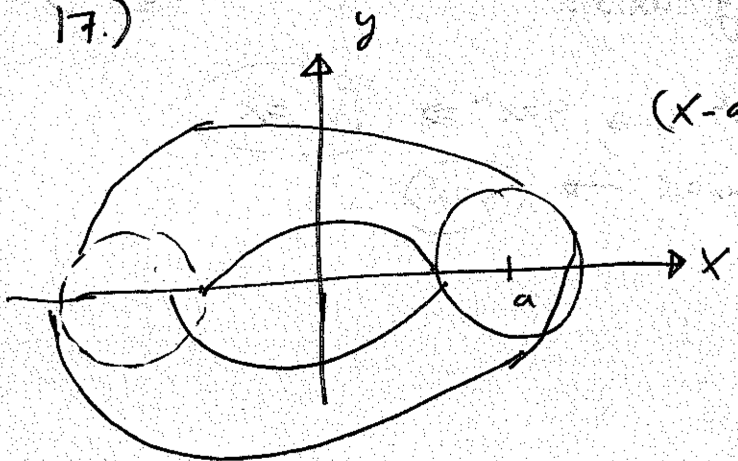
where: $dV = \pi y^2 d\cancel{x}$

$$= \pi\left[1 - x^2\right] dx$$

$$\therefore V = \int_{-1}^{1} dx\, \pi\left[1 - x^2\right] = \pi\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \pi\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$$

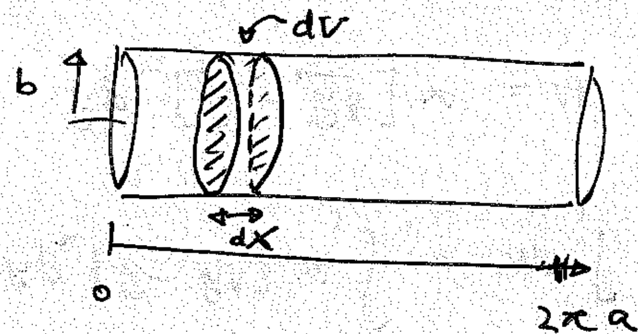
$$= \boxed{\frac{4}{3}\pi}$$

17.)



$(x-a)^2 + y^2 = b^2 \qquad \underline{(a>b)}$

← "unrolling" this gives us the following:

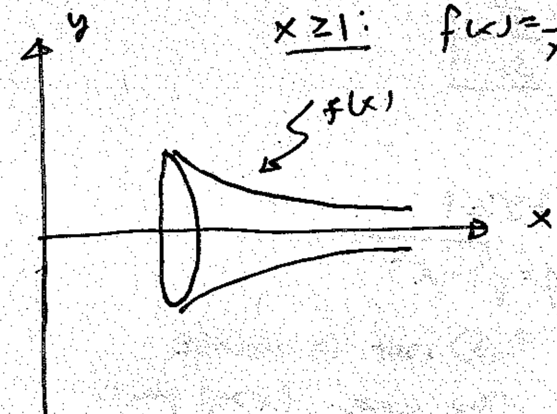


$dV = \pi b^2 dx$

so: $V = \Sigma\, dV$

$= \int \pi b^2 dx = \pi b^2 2\pi a \longrightarrow \boxed{V = 2\pi^2 a b^2}$

18.) $\underline{x \geq 1}$: $f(x) = \frac{1}{x}$



(a) Volume:

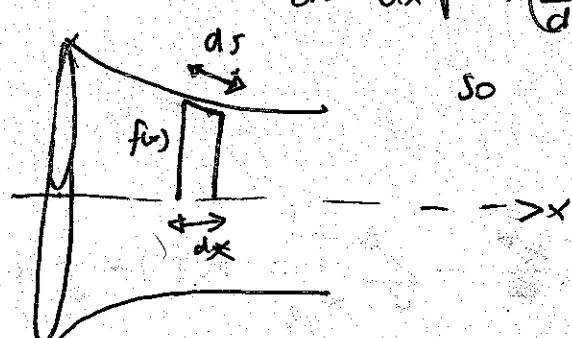
$$V = \int_1^\infty \pi [f(x)]^2 dx$$

$$= \int_1^\infty \frac{\pi}{x^2} dx$$

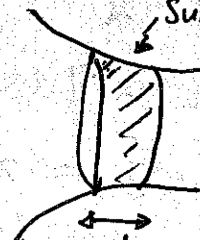
$$= \left.\frac{-\pi}{x}\right|_1^\infty = \boxed{\pi}$$

(b) Surface area:

$ds = dx\sqrt{1+\left(\frac{df}{dx}\right)^2}$



so



Surface area of this piece

$= 2\pi f(x) \cdot ds$

$= \frac{2\pi}{x}\sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx$

$= \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\, dx$

$= dA$

So ∴ total S.A. is:

$$A = \sum dA$$

$$= \int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx$$

Notice that as $x \to \infty$: $\frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}$

$$\sim \frac{2\pi}{x}\sqrt{1}$$

so : for x_0 very large,

$$\int_{x_0}^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx$$

$$\approx \int_{x_0}^\infty \frac{2\pi}{x}\,dx = \underbrace{2\pi \ln(x)}\Big|_{x_0}^\infty$$

blows up since $\lim_{x\to\infty} \ln(x) = \infty$

From this, we can see why $\int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx$

blows up (i.e. $\to +\infty$)

Since

$$\int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx > \int_{x_0}^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx \to \infty.$$

$(x_0 > 1)$

$\boxed{\therefore \text{ Surface area is infinite}}$

$\boxed{(c)}$ Notice that the trumpet extends out to infinity
($x \to \infty$).

~~So to coat both the surface~~

So even though the trumpet has a finite volume (π), to fill up all of the trumpet, paint has to be extended out to $x \to +\infty$. As the paint spreads towards larger and larger position x, larger and larger portion of the π-volume of trumpet gets to be filled up.
But to get exactly all of π-volume, paint has to get out to ∞.
(which is to say that this is a limiting process.)
To coat the surface with the paint, the same is true.
For both the volume & the surface, as x gets larger and larger, less and less vol. & surface area is added to the trumpet. It's just that while the volume approaches a finite volume π, surface area does not.
But the bottom line is that in order to fill up the entire trumpet, paint has to travel out to $x = +\infty$.
This can never happen in reality.

19) We derived the formula in Class notes.

- For $f(x) = \frac{x^3}{3} + 47\pi$ for $x \in [0, \pi/4]$, we have:

$$\text{Length} = \int_0^{\pi/4} dx \sqrt{1 + \frac{df}{dx}}$$

$$= \int_0^{\pi/4} dx \sqrt{1 + x^2}$$

let $x = \tan u$

$dx = \sec^2 u \, du$

$$= \int_{u_a}^{u_b} \sec^3 u \, du$$

$\Rightarrow 0 = \tan u_a \Rightarrow u_a = 0$

$\frac{\pi}{4} = \tan u_b \Rightarrow u_b = \text{Arctan}\left(\frac{\pi}{4}\right)$

$$= \boxed{\int_0^{\text{Arctan}(\pi/4)} \sec^3 u \, du}$$

↑ you can leave your answer in this form (Hard integral to do.)

20) (i) $f(x) = e^{e^x}$: degree 3, at 0.

so: $f(0) = e$

$$\left.\frac{df}{dx}\right|_{x=0} = \left. e^{e^x} e^x \right|_{x=0} = e$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=0} = \left.\left(e^x e^{e^x} + e^x \left[e^{e^x} e^x \right] \right)\right|_{x=0} = e + e = 2e$$

$$\left.\frac{d^3f}{dx^3}\right|_{x=0} = \left. e^x e^{e^x} + e^x\left[e^{e^x} e^x\right] + e^x\left[e^{e^x} e^x\right] + e^x\left[\left.\frac{d^2f}{dx^2}\right|_{x=0}\right]\right|_{x=0}$$

$$= e + e + e + 2e = 5e$$

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$\therefore$

$$f(x) = e^{x} = e + e\cdot(x) + \frac{2e\,x^2}{2} + \frac{5e\,x^3}{3!}$$

$$= \boxed{e + xe + x^2 e + \frac{5x^3}{6} e}$$

(ii) $f(x) = e^{\sin x}$; deg. 3 at 0.

$f(0) = 1$

$$\left.\frac{df}{dx}\right|_{x=0} = e^{\sin x} \cos x \Big|_{x=0} = 1$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=0} = e^{\sin x} \cos^2 x - e^{\sin x} \sin x \Big|_{x=0} = 1$$

$$\left.\frac{d^3 f}{dx^3}\right|_{x=0} = e^{\sin x} \cos^3 x - 2e^{\sin x} \cos x \sin x - e^{\sin x} \sin x \cos x - e^{\sin x} \cos x \Big|_{x=0}$$

$$= 1 - 1 = 0$$

$$\therefore\ f(x) = e^{\sin x} = 1 + x + \frac{x^2}{2} + \cancel{\frac{0 \cdot x^3}{3!}} \to 0$$

$$= \boxed{1 + x + \frac{x^2}{2}}$$

(iii)

$\sin x = f(x)$

$f(\pi/2) = 1$

$f'(\pi/2) = \cos(\pi/2) = 0$

$f''(\pi/2) = -\sin x \Big|_{\pi/2} = -1$

$f^{(3)}(\pi/2) = -\cos x \Big|_{\pi/2} = 0.$

$f^{(4)}(\pi/2) = \sin x \Big|_{\pi/2} = 1$

and start all over again.

i.e., $f^{(1)} = f^{(5)}$

$f^{(2)} = f^{(6)}$

$f^{(3)} = f^{(7)}$ and so on...

So:

$$f(x) = \sin(x)$$
$$= 1 - \frac{(x-\pi/2)^2}{\underbrace{2!}_{(2\times1)!}} + \frac{(x-\pi/2)^4}{\underbrace{4!}_{(2\times2)!}} - \frac{(x-\pi/2)^6}{\underbrace{6!}_{(2\cdot3)!}} + \dots$$

$$= \boxed{1 + \sum_{i=1}^{n} \frac{(-1)^i (x-\pi/2)^{2i}}{(2i)!}}$$

(iv) $f(x) = \cos x$; degree $2n$, at π.

$f^{(0)}(\pi) = -1$

$f^{(1)}(\pi) = -\sin(\pi) = 0$

$f^{(2)}(\pi) = -\cos(\pi) = 1$

$f^{(3)}(\pi) = \sin(\pi) = 0$

$f^{(4)}(\pi) = \cos(\pi) = f^{(0)}(\pi) = -1$

$f^{(5)}(\pi) = f^{(1)}(\pi)$ and so on...

$f^{(6)}(\pi) = f^{(2)}(\pi)$

$f^{(7)}(\pi) = f^{(3)}(\pi)$

So: $f(x) = \cos x$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} + \dots$$

$$= \boxed{-1 + \sum_{i=1}^{n} \frac{(x-\pi)^{2i}}{(2i)!}(-1)^{i+1}}$$

(v) $f(x) = e^x$; degree n, at 2.

~~etc~~

$f(2) = e^2$

$f'(2) = e^2$

$f''(2) = e^2$

and so on.

$f^{(n)}(2) = e^2$

for <u>all</u> n.

so:

$$e^x = e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \dots\right]$$

$$= \boxed{e^2 \sum_{i=0}^{n} \frac{(x-2)^i}{i!}}$$

(vi) $f(x) = \frac{1}{1+x^2}$; degree $2n+1$, at 0.

$f(0) = 1$

$f'(0) = \frac{-1}{(1+x^2)^2} 2x \Big|_{x=0} = 0$

$f^{(2)}(0) = \frac{-2(1+x^2)^2 + 4x(1+x^2)2x}{(1+x^2)^4} \Big|_{x=0} = -2$

$f^{(3)}(0) = 0$

$f^{(4)}(0) = 4!$ ... and so on.

so: $\boxed{\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots + (-1)^n x^{2n}}$

21.) (i) $\sin x = x - \underbrace{\frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}$ ← Taylor series of $\sin x$.

these higher order terms, $x^3, x^5, x^7, \ldots$ are much smaller than x when ~~$\ldots$~~ $|x| \ll 1$.

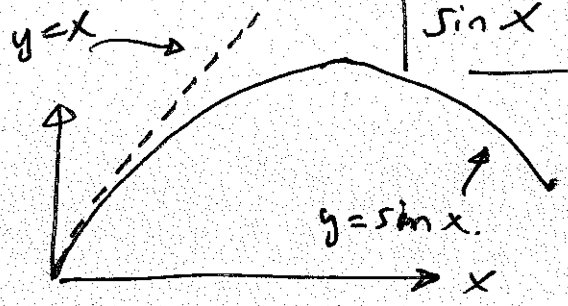
To see this, note that $\frac{x^m}{x^n} = x^{m-n}$
if $m > n$

so $\lim_{x \to 0} x^{m-n} = 0$.

so: good approx of $\sin x$ is:

$$\boxed{\sin x \approx x} + \cdots$$

Ignoring these small terms.



(ii) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$ ← Taylor series of $\cos x$.

Again, for the same reason as above, (in (i)) $\boxed{\cos x \approx 1 - \frac{x^2}{2}.}$

when x close to 0

(iii) $\tan x = \frac{\sin x}{\cos x} \approx \frac{x}{\left(1-\frac{x^2}{2}\right)}$ ← from (i) ← from (ii)

but when $x \approx 0$: $\frac{1}{1-\frac{x^2}{2}} \approx 1 + \frac{x^2}{2}$

↑ by taylor series of this

so: $\tan x \approx x \cdot \left[1 + \frac{x^2}{2}\right]$

$= x + \underbrace{\frac{x^3}{2}}$ ← much smaller than x when x close to 0.

$\approx x$

so: $\boxed{\tan x \approx x}$