MITES 09 - Calculus II - Solution Set 3

Solution 1.:

$$\begin{aligned}
\frac{dz}{dx} + 5x &= 0 \\
\Rightarrow dz &= -5x dx \\
\Rightarrow \int_{z_0}^{z_f} dz &= \int_{x_0}^{x_f} -5x dx \quad \text{(where, } z_0 = z(x_0)\text{, and } z_f = z(x_f)\text{)} \\
\Rightarrow z_f - z_0 &= \frac{-5x_f^2}{2} + \frac{5x_0^2}{2} \\
\Rightarrow z(x) &= \frac{-5x^2}{2} + z_0 + \frac{5x_0^2}{2} \quad \text{(where, we relabeled } z = z_f \text{ and } x = x_f\text{)} \\
\Rightarrow z(x) &= \frac{-5x^2}{2} + c \quad \text{(where, we relabeled constant c} = z_0 + \frac{5x_0^2}{2}\text{)}
\end{aligned}$$

Thus, the solution to our differential equation is: $\boxed{z(x) = \frac{-5x^2}{2} + c}$.

Solution 2.:

$$\begin{aligned}
\frac{dy}{dt} + at^2 + b &= 0 \\
\Rightarrow dy &= (-at^2 - b)dt \\
\Rightarrow \int_{y_0}^{y_f} dy &= \int_{t_0}^{t_f} (-at^2 - b)dt \quad \text{(where, } y_0 = y(t_0)\text{, and } y_f = y(t_f)\text{)} \\
\Rightarrow y_f - y_0 &= \frac{-at_f^3}{3} - bt_f + \frac{at_0^3}{3} + bt_0 \\
\Rightarrow y(t) &= -\frac{at^3}{3} - bt + y_0 + \frac{at_0^3}{3} + bt_0 \quad \text{(where, we relabeled } y = y_f \text{ and } t = t_f\text{)} \\
\Rightarrow y(t) &= -\frac{at^3}{3} - bt + c. \quad \text{(where, we relabeled constant c} = y_0 + \frac{at_0^3}{3} + bt_0\text{)}
\end{aligned}$$

Thus, the solution to our differential equation is: $\boxed{y(t) = -\frac{at^3}{3} - bt + c}$.

Solution 3.:

$$\begin{aligned}
\frac{dx}{dz} &= z - e^z \\
\Rightarrow dx &= (z - e^z)dz \\
\Rightarrow \int_{x_0}^{x_f} dx &= \int_{z_0}^{z_f} (z - e^z)dz \quad \text{(where, } x_0 = x(z_0)\text{, and } x_f = x(z_f)\text{)} \\
\Rightarrow x_f - x_0 &= -exp(z_f) + \frac{z_f^2}{2} + exp(z_0) - \frac{z_0^2}{2} \\
\Rightarrow x(z) &= -exp(z) + \frac{z^2}{2} + x_0 + exp(z_0) - \frac{z_0^2}{2} \quad \text{(where, we relabeled } x = x_f \text{ and } z = z_f\text{)} \\
\Rightarrow x(z) &= -exp(z) + \frac{z^2}{2} + c. \quad \text{(where, we relabeled constant c} = x_0 + exp(z_0) - \frac{z_0^2}{2}\text{)}
\end{aligned}$$

Thus, the solution to our differential equation is: $\boxed{x(z) = -exp(z) + \frac{z^2}{2} + c}$.

Solution 4.:

$$\begin{aligned}
\frac{dx}{dt} &= 3 - \frac{1}{t} \\
\Rightarrow dx &= (3 - \frac{1}{t})dt \\
\Rightarrow \int_{x_0}^{x_f} dx &= \int_{t_0}^{t_f} (3 - \frac{1}{t})dt \quad \text{(where, } x_0 = x(t_0)\text{, and } x_f = x(t_f)\text{)} \\
\Rightarrow x_f - x_0 &= (3t - ln(t))\Big|_{t_0}^{t_f} \\
\Rightarrow x_f &= 3t_f - ln(t_f) - 3t_0 + ln(t_0) + x_0 \\
\Rightarrow x(t) &= 3t - ln(t) + x_0 - 3t_0 + ln(t_0) \quad \text{(where, we relabeled } x = x_f \text{ and } t = t_f\text{)} \\
\Rightarrow x(t) &= 3t - ln(t) + c \quad \text{(where, we relabeled constant c} = x_0 - 3t_0 + ln(t_0)\text{)}
\end{aligned}$$

Thus, the solution to our differential equation is: $\boxed{x(t) = 3t - ln(t) + c}$. Notice that we must have $t > 0$ since ln(t) is an undefined function for $t < 0$

Solution 5.:

$$\begin{aligned}
x\frac{dx}{dt} &= -bt \\
\Rightarrow xdx &= -btdt \\
\Rightarrow \int_{x_0}^{x_f} xdx &= \int_{t_0}^{t_f} -btdt \text{(where, } x_0 = x(t_0)\text{, and } x_f = x(t_f)\text{)} \\
\Rightarrow \frac{x_f^2}{2} - \frac{x_0^2}{2} &= -\frac{bt_f^2}{2} + \frac{bt_0^2}{2} \\
\Rightarrow [x(t)]^2 &= x_0^2 + bt_0^2 - bt^2 \quad \text{(where, we relabeled } x = x_f \text{ and } t = t_f\text{)} \\
\Rightarrow [x(t)]^2 &= c - bt^2 \quad \text{(where, we relabeled constant c} = x_0^2 + bt_0^2\text{)} \\
\Rightarrow x(t) &= \pm\sqrt{c - bt^2}
\end{aligned}$$

Thus, the solution to our differential equation is: $\boxed{x(t) = \pm\sqrt{c - bt^2}}$.

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Solution 6.:

$$
\begin{aligned}
\frac{dk}{dt} &= 3-k \\
\Rightarrow \frac{dk}{3-k} &= dt \\
\Rightarrow \int_{k_0}^{k_f} \frac{dk}{3-k} &= \int_{t_0}^{t_f} dt \quad \text{(where, } k_0 = k(t_0)\text{, and } k_f = k(t_f)\text{)} \\
\Rightarrow -ln(3-k)\Big|_{k_0}^{k_f} &= t_f - t_0 \\
\Rightarrow -ln(3-k_f) + ln(3-k_0) &= t_f - t_0 \\
\Rightarrow ln(\frac{1}{3-k_f}) &= (t_f - t_0) - ln(3-k_0) \\
\Rightarrow exp\left[ln(\frac{1}{3-k_f})\right] &= exp\left[(t_f - t_0) - ln(3-k_0)\right] \\
\Rightarrow \frac{1}{3-k_f} &= exp\left[t_f - t_0\right] exp\left[-ln(3-k_0)\right] \\
\Rightarrow \frac{1}{3-k_f} &= \frac{exp\left[t_f - t_0\right]}{3-k_0} \\
\Rightarrow \frac{3-k_0}{exp\left[t_f - t_0\right]} &= 3-k_f \\
\Rightarrow k(t) &= 3-(3-k_0)exp(-t)exp(t_0) \quad \text{(where, we relabeled } k = k_f \text{ and } t = t_f\text{)} \\
\Rightarrow k(t) &= 3 - ce^{-t} \quad \text{(where, we relabeled constant c} = (3-k_0)exp(t_0)\text{)}
\end{aligned}
$$

Thus, the solution to our differential equation is: $\boxed{k(t) = 3 - ce^{-t}}$.