Solution set # 4

1.)

1(a) Let $y_1 = e^{2x}$ $y_2 = e^{3x}$.

Then $y_1'' - 5y_1' + 6y_1$
$= 4e^{2x} - 5\cdot 2e^{2x} + 6e^{2x}$
$= -6e^{2x} + 6e^{2x}$
$= 0$

and $y_2'' - 5y_2' + 6y_2$
$= 9e^{3x} - 15e^{3x} + 6e^{3x}$
$= -6e^{3x} + 6e^{3x}$
$= 0.$

So Both $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are solutions of the differential eq'n : $y'' - 5y' + 6y = 0$

1(b) We know from (a) that $y_1 = e^{2x}$ and $y_2 = e^{3x}$ are both solutions. Now, consider $z_1 = c_1 y_1$ and $z_2 = c_2 y_2$.
(where c_1 and c_2 are two arbitrary constants.)

Then, $z_1'' - 5z_1' + 6z_1$
$= c_1 \cdot \underbrace{(y_1'' - 5y_1' + 6y_1)}_{= 0 \text{ (from (a))}}$
$= 0$

and $z_2'' - 5z_2' + 6z_2$
$= c_2 \times \underbrace{(y_2'' - 5y_2' + 6y_2)}_{= 0 \text{ (from (a))}}$
$= 0$

So $z_1(x)$ and $z_2(x)$ are both solutions.
(True for any values of c_1 and c_2.)

And note that for $y(x) = z_1(x) + z_2(x)$
$= c_1 e^{2x} + c_2 e^{3x}$, we have

$y'' - 5y' + 6y$
$= (z_1 + z_2)'' - 5(z_1 + z_2)' + 6(z_1 + z_2)$
$= \underbrace{(z_1'' - 5z_1' + 6z_1)}_{= 0} + \underbrace{(z_2'' - 5z_2' + 6z_2)}_{= 0}$
$= 0.$

Hence $y(x)$ is a solution of the equation for any values of c_1 & c_2.

$\boxed{(c)}$ We have already shown in (b) that the differential eqn is a <u>linear equation</u> (since $\underbrace{z_1(x)}_{\text{solution}} + \underbrace{z_2(x)}_{\text{solution}} = \underbrace{y(x)}_{\text{yields another solution}}$)

$\boxed{(2)}$ $y(x) = e^{mx}$.

Then
$$2y''' + y'' - 5y' + 2y$$
$$= 2m^3 y(x) + m^2 y(x) - 5m y(x) + 2y(x)$$
$$= \left[2m^3 + m^2 - 5m + 2\right] y(x)$$
$$= 0$$

$\Rightarrow$ Need $\underline{2m^3 + m^2 - 5m + 2 = 0}$ (because $y(x) \neq 0$ for all values of x.)

To find all 3 roots (i.e., 3 values of m satisfying above eqn),

we first guess one of the 3 values.

<u>Let $m = 1$</u>: Then notice that
$$2(1)^3 + (1)^2 - 5(1) + 2$$
$$= 2 + 1 - 5 + 2$$
$$= -2 + 2$$
$$= 0 \quad ✓ \quad \Rightarrow \quad m = 1 \text{ indeed is one of the 3 roots.}$$

So: $0 = 2m^3 + m^2 - 5m + 2$

$$\Rightarrow 0 = \underbrace{(m-1)}_{\substack{\text{Since } m=1 \text{ is} \\ \text{one of the roots.}}} \underbrace{(am^2 + bm + c)}_{\text{Need to find values of } a, b, c.}$$

Need, $am^3 = 2 \Rightarrow a=2$

$m^2 = bm^2 - am^2$
$= bm^2 - 2m^2 \Rightarrow b=3$

and, $-c = 2 \Rightarrow c=-2$

so: $0 = (m-1)\underline{(2m^2+3m-2)}$

$\overset{\uparrow}{\underset{0}{\shortparallel}}$ if $m_{\pm} = \dfrac{-3 \pm \sqrt{9-4(2)(-2)}}{4}$

$= \dfrac{-3 \pm \sqrt{25}}{4}$

$\Rightarrow m_+ = \frac{1}{2}$ $m_- = -2$

so: $\boxed{m_1 = 1 \,, \; m_2 = \frac{1}{2} \,, \; m_3 = -2}$

↑ 3 possible values of m for which $y(x) = e^{mx}$ is a solution.

~~From~~ Our differential eq'n is a linear eq'n

so: the most general ~~soln~~ sol'n to our eq'n is:

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

$$\Rightarrow \quad = \boxed{c_1 e^{x} + c_2 e^{x/2} + c_3 e^{-2x}}$$

(c_1, c_2, c_3 are arbitrary constants)

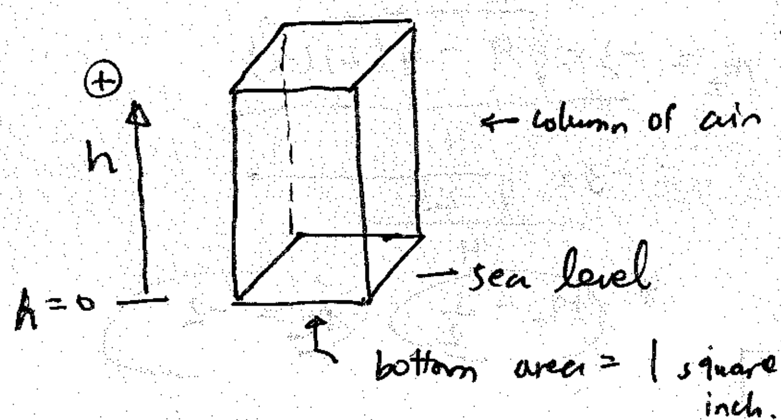
~~This~~ our differential eq'n is of order 3

(i.e., 3^{rd} derivative is the highest ~~order~~ derivative that appears in the eq'n).

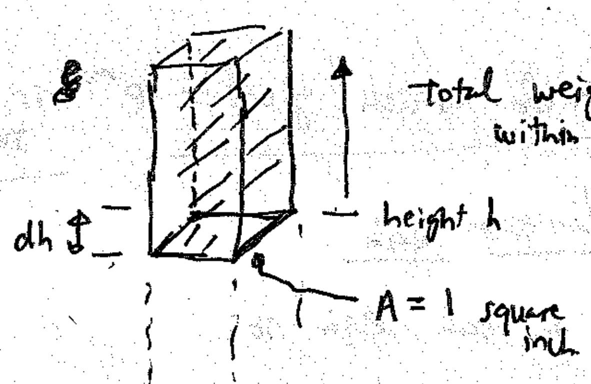
So by the general theorem we learned in class, there ~~must~~ ~~be~~ has to be exactly 3 arbitrary constants (c_1, c_2, c_3) in the most general sol'n.

So there cannot be a fourth constant "C_4" in our solution. (If we could have a "C_4", then you should have found four possible values of m, <u>not</u> 3.).

$\boxed{3.)}$



$\rho(h)$ = density of air at height h
$= a\,p(h)$ $\quad a$ = proportionality constant.
$p(h)$: pressure at height h.
$\because$ We're told that density is proportional to pressure.



Total weight of air within the column above height h is:

$$(\rho(h)\,\underset{1}{\underset{\shortparallel}{A}}\,dh)\,g = \rho(h)\,g\,dh$$

$$\Rightarrow \text{Total weight} = \int_h^{\infty} \rho(h) g\, dh$$
$$= \int_h^{\infty} a\,p(h)\,g\,dh$$
$$= -\int_{\infty}^{h} (ag)\,p(h)\,dh$$

~~Pressure at height h~~

But, $\displaystyle p(h) = \frac{\text{Total weight}}{A}$

$$\Rightarrow \quad p(h) = -\int_{\infty}^{h} (ag)\,p(h)\,dh$$

$$\Rightarrow \quad \frac{dp}{dh} = -ag\,p(h)$$ ← By <u>fundamental thm. of calculus</u>. (discussed in class before midterm exam.)

so, letting $c = ag$

& constant.

we have $\boxed{\frac{dp}{dh} = -cp}$

Solve:
$$\int_{P(h=0)}^{P(h)} \frac{dp}{p} = \int_{h=0}^{h} -c\,dh$$

$$\Rightarrow \ln\left(\frac{P(h)}{P(h=0)}\right) = -ch$$ call $P_0 = P(h=0)$

$$\Rightarrow \boxed{p(h) = P_0 e^{-ch}}$$ & pressure at sea level.

4.) (a) $\frac{d^2y}{dt^2} = -g.$; $y = y(t)$

$V(t) = \frac{dy}{dt} \Rightarrow \frac{dV}{dt} = -g$

$$\Rightarrow \int_{V(t=0)}^{V(t)} dV = \int_{t=0}^{t} -g\,dt$$

$$\Rightarrow V(t) - V(t=0) = -gt$$

$$\Rightarrow \boxed{V(t) = V_0 - gt}$$ V_0 = velocity at time $t=0$ (Initial velocity)

(b)
$$\frac{dy}{dt} = v_0 - gt$$

$$\Rightarrow \int_{y(t=0)}^{y(t)} dy = \int_{t=0}^{t} (V_0 - gt)\,dt \Rightarrow y(t) - y(t=0) = V_0 t - \frac{gt^2}{2}$$

$$\Rightarrow \boxed{y(t) = y_0 + V_0 t - \frac{gt^2}{2}}$$

(c)

$$\frac{dv}{dt} = -g - cv^2 \qquad c = k/m \leftarrow \text{constants}$$

$$g$$

$$\Rightarrow \int_{V(t=0)}^{V(t)} \frac{dV}{-g - cV^2} = \int_{t=0}^{t} dt$$

Let $V(t=0) = V_0$.

$$\Rightarrow -\frac{1}{g} \int_{V_0}^{V(t)} \frac{dV}{1 + \left(\frac{c}{g}\right) V^2} = t$$

Let $u = \sqrt{\frac{c}{g}}\, V$

$\Rightarrow du = \sqrt{\frac{c}{g}}\, dv$; $V(t) = \sqrt{\frac{g}{c}}\, u(t)$

$V_0 = \sqrt{\frac{g}{c}}\, u_0$

$$\Rightarrow t = -\frac{1}{g} \int_{u_0}^{u(t)} \frac{\sqrt{\frac{g}{c}}\, du}{1 + u^2}$$

$$= -\frac{1}{\sqrt{gc}} \int_{u_0}^{u(t)} \frac{du}{1+u^2} \leftarrow \text{Let } u = \tan\theta$$

$\Rightarrow du = \sec^2\theta\, d\theta$

$(1 + u^2 = 1 + \tan^2\theta = \sec^2\theta)$

$$= -\frac{1}{\sqrt{gc}} \int_{\theta_0}^{\theta(t)} \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{-1}{\sqrt{gc}} \int_{\theta_0}^{\theta(t)} d\theta$$

$$= \frac{-1}{\sqrt{gc}} \left[\theta(t) - \theta_0\right] \leftarrow$$

$\theta(t) = \text{Arctan}(u(t))$

$= \text{Arctan}\left(\sqrt{\frac{c}{g}}\, V(t)\right)$

$\theta_0 = \text{Arctan}\left(\sqrt{\frac{c}{g}}\, V_0\right)$

$$t = \frac{-1}{\sqrt{gc}} \left[\text{Arctan}\left(\sqrt{\frac{c}{g}}\, V(t)\right) - \text{Arctan}\left(\sqrt{\frac{c}{g}}\, V_0\right)\right]$$

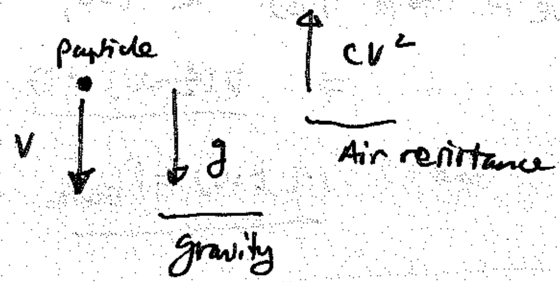
Now, we're told that $V_0 = V(t=0) = 0$ $\Rightarrow \text{Arctan}\left(\sqrt{\frac{c}{g}}\, V_0\right) = 0$.

$$\Rightarrow t = -\frac{1}{\sqrt{gc}} \text{Arctan}\left(\sqrt{\frac{c}{g}}\, V(t)\right)$$

$$\Rightarrow \boxed{V(t) = \sqrt{\frac{g}{c}} \tan\left(-\sqrt{gc}\, t\right)}$$

To describe a falling object with air resistance, we need to change the sign in front of g.

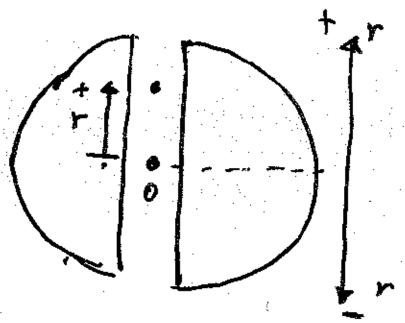
$\Rightarrow \quad \dfrac{dV}{dt} = g - cV^2$ since



Terminal velocity achieved when $\dfrac{dV}{dt} = 0$:

$\Rightarrow \quad g - cV^2 \Rightarrow \boxed{V_t = \sqrt{\dfrac{g}{c}}}$ ← terminal velocity (Attained after long enough time.)

$\boxed{5.)}$



$\underline{0}$: center of Earth.

$\underline{r}$: position of particle relative to center of the Earth.

$F_{gravity} = -cr \qquad \underline{c > 0}$

(since within Earth, gravity is proportional to distance from center of the Earth.)

So: $\quad m\dfrac{d^2r}{dt^2} = -cr$ ← Newton's 2^{nd} law. $\qquad m =$ mass of particle

$\Rightarrow \quad \dfrac{d^2r}{dt^2} + \dfrac{c}{m}r = 0$ ← Eq'n we solved in class.

Solve by guessing : $r_1(t) = A\cos\left(\sqrt{\frac{c}{m}}\,t\right)$, $\quad r_2(t) = B\sin\left(\sqrt{\frac{c}{m}}\,t\right)$ ← you can check both are solutions for arbitrary constants A & B.

And $r(t) = r_1(t) + r_2(t)$ ← general sol'n containing 2 constants

$= A\cos\left(\sqrt{\frac{c}{m}}\,t\right) + B\sin\left(\sqrt{\frac{c}{m}}\,t\right)$

To find A & B: use the initial ($t=0$) condition:

We're told that at $t=0$, the rock is simply dropped.

$$\Rightarrow \quad V(t=0)=0 \quad \text{--- ①}$$

And $r(t=0) = R_{Earth}$ ← radius of Earth --- ②

So: $r(t=0) = A = R_{Earth}$

And $\dot{r}(t) = V(t) = -\sqrt{\frac{c}{m}}\, A \sin\left(\sqrt{\frac{c}{m}}\, t\right) + \sqrt{\frac{c}{m}}\, B \cos\left(\sqrt{\frac{c}{m}}\, t\right)$

$$\Rightarrow \quad V(0) = 0 = \sqrt{\frac{c}{m}}\, B \quad \Rightarrow \quad B = 0$$

$$\therefore \quad \boxed{r(t) = R_{Earth} \cos\left(\sqrt{\frac{c}{m}}\, t\right)}$$

$$\Downarrow$$

$$\boxed{V(t) = -\sqrt{\frac{c}{m}}\, R_{Earth} \sin\left(\sqrt{\frac{c}{m}}\, t\right)}$$

So $0 = r(t_f)$ $\quad t_f \equiv$ time at which center of Earth reached.

$$\Rightarrow \quad 0 = \cos\left(\sqrt{\frac{c}{m}}\, t_f\right) \Rightarrow t_f \sqrt{\frac{c}{m}} = \frac{\pi}{2}$$ ← first t_f at which center is reached

$$\Rightarrow \quad t_f = \frac{\pi}{2}\sqrt{\frac{m}{c}}$$

So: $$V(t_f) = -\sqrt{\frac{c}{m}}\, R_{Earth} \underbrace{\sin\left(\sqrt{\frac{c}{m}}\sqrt{\frac{m}{c}}\,\frac{\pi}{2}\right)}_{=1}$$

$$= \boxed{-\sqrt{\frac{c}{m}}\, R_{Earth}}$$ ← velocity of rock when it's at the center of the Earth.

⊖ since rock is moving in ⊖ direction.

$\Rightarrow$ ~~speed~~

$\boxed{6.)}$

$\boxed{(a)}$ $y'' - y' - 2y = 0$

Guess : $y(x) = e^{\alpha x}$

Differential Eq'n becomes $\Rightarrow$ $\alpha^2 - \alpha - 2 = 0 \Rightarrow \alpha_{\pm} = \frac{1 \pm \sqrt{1+8}}{2}$

$\Rightarrow \alpha_+ = 2 \quad \alpha_- = -1$

$\therefore$ General solution : $\boxed{y(x) = c_1 e^{2x} + c_2 e^{-x}}$

$\boxed{(b)}$ Let $y_p(x) = ax + b$.

Then plugging into $y_p'' - y_p' - 2y_p = 4x$ we get :

$$-a - 2(ax+b) = 4x$$

$$\Rightarrow -2ax - a - 2b = 4x \Rightarrow -2a = 4 \quad \ldots (1)$$

and $-a - 2b = 0 \quad \ldots (2)$

So $\boxed{a = -2}$ and $\boxed{b = 1}$ $\Rightarrow$ $y_p(x) = -2x + 1$ is a particular solution to differential eq'n.

So, General sol'n to our differential eq'n is :

$$y(x) = y_p(x) + c_1 e^{2x} + c_2 e^{-x}$$

$$= \boxed{c_1 e^{2x} + c_2 e^{-x} - 2x + 1}$$

7.) **(a)** $x^3y'' + x^2y' + xy = 1.$

Guess: $y_p(x) = ax^n$ $\quad \left(\begin{array}{l} a=? \\ n=? \end{array}\right)$

Check: $a\left\{x^3 n(n-1)x^{n-2} + x^2 n x^{n-1} + x x^n\right\} = 1$

$\Rightarrow \quad a\{n(n-1)x^{n+1} + n x^{n+1} + x^{n+1}\} = 1$

$\Rightarrow \quad a[n(n-1) + n + 1]\, x^{n+1} = 1$

So we can satisfy above eq'n by first letting $n = -1$

(so $x^{n+1} = x^{-1+1} = x^0 = 1$) (No x!)

$\Rightarrow$

$a[-1(-2) - 1 + 1] = 1$

$\Rightarrow \boxed{a = \frac{1}{2}} \Rightarrow \boxed{y_p(x) = \frac{1}{2}x^n}$

↑ particular sol'n

(But not general solution)

Not a linear eq'n

(b) $y'' - 2y' = 6$

Guess: $y_p(x) = ax$

$\Rightarrow \quad y_p'' - 2y_p' = 6$

$\Rightarrow \quad -2a = 6 \Rightarrow a = -3 \Rightarrow \boxed{y_p(x) = -3x}$ ← particular solution

(But not general solution)

Eq'n is linear

$\boxed{1(c)}$ $y'' - 2y = \sin(x)$

Guess : $y_p(x) = a\sin(x)$

$\Rightarrow \quad y_p'' - 2y_p = \sin(x)$

$\Rightarrow \quad -a\cancel{\sin(x)} - 2a\cancel{\sin(x)} = \cancel{\sin(x)}$

$\Rightarrow \quad -3a = 1 \Rightarrow \boxed{a = -\frac{1}{3}}$ $\Rightarrow$ $\boxed{y_p(x) = -\frac{1}{3}\sin(x)}$

particular soln
(But not general solution)

$\boxed{\text{(Eqn is linear)}}$

$\boxed{8.)}$

$\boxed{1(a)}$ $y'' + a^2 y = \sin(bx)$ .... (3)

can be thought of as made up of 2 eqns. :

$y_G'' + a^2 y_G = 0$ ...... (1)

$y_p'' + a^2 y_p = \sin(bx)$ ... (2)

So (3) is : Eqn (1) + Eqn (2) = Eqn (3)

i.e. $\underbrace{(y_G + y_p)''}_{y''} + a^2 \underbrace{(y_G + y_p)}_{y} = \sin(bx)$

Solution to eqn (1) : $\boxed{y_G(x) = A\sin(ax) + B\cos(ax)}$

Solution to eqn (2) : Guess : $y_p(x) = C\sin(bx)$

then check : $-b^2 C\cancel{\sin(bx)} + a^2 C\cancel{\sin(bx)} = \cancel{\sin(bx)}$

$\Rightarrow \quad C = \dfrac{1}{a^2 - b^2}$

So $\boxed{y_p(x) = \frac{1}{a^2-b^2}\sin(bx)}$ ← soln to eqn ②

Thus, $y(x) = y_G(x) + y_p(x)$

$$\Rightarrow \boxed{y(x) = A\sin(ax) + B\cos(ax) + \frac{1}{a^2-b^2}\sin(bx)}$$

← THE solution to the differential eqn

(b)

$$\boxed{y(x) = y_G(x) + \frac{3}{a^2-4}\sin(2x) + \frac{5}{a^2-100}\sin(10x) + \frac{90\sin(3x)}{a^2-9} + \frac{100\sin(101x)}{a^2-101^2}}$$

← solution (obtained by linearity)

where $y_G(x) = A\sin(ax) + B\cos(ax)$

(c)

Most general soln:

$$\boxed{y(x) = y_G(x) + \sum_{n=1}^{\infty}\frac{b_n\sin(nx)}{a^2-n^2}}$$

Assuming $(a \neq n)$

(d) Most general soln:

$$\boxed{y(x) = y_G(x) + \sum_{n=1}^{\infty}\left\{\frac{a_n\cos(nx) + b_n\sin(nx)}{a^2-n^2}\right\}}$$

Assuming $(a \neq n)$

9.)

(a) $u(x) = \sin(nx)$; $v(x) = \cos(mx)$.

Then straight forward to show that : $u'' + n^2 u = 0$

$v'' + m^2 v = 0$

(b) $$\int_{-\pi}^{\pi} u'' v \, dx = \underline{u'(x) v(x) \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} u'(x) v'(x) \, dx$$

$$= \underline{n \cos(nx) \cos(mx) \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} u'(x) v'(x) \, dx$$

$$\| $$

$$n \begin{pmatrix} \cos(n\pi) \cos(m\pi) \\ - \cos(-n\pi) \cos(-m\pi) \end{pmatrix}$$

same since $\cos(nx)\cos(mx)$ is even function

So $$\boxed{u'(x) v(x) \Big|_{-\pi}^{\pi} = 0}$$

$$\therefore \boxed{\int_{-\pi}^{\pi} u'' v \, dx = - \int_{-\pi}^{\pi} u'(x) v'(x) dx}$$

Similarly, $$\boxed{\int_{-\pi}^{\pi} u(x) v''(x) dx = - \int_{-\pi}^{\pi} u'(x) v'(x) dx}$$

(c) $$\int_{-\pi}^{\pi} u''(x) v(x) \, dx = \int_{-\pi}^{\pi} (-n^2 u) v dx$$ since $u'' + n^2 u = 0$

$$= -n^2 \int_{-\pi}^{\pi} u v \, dx.$$

Similarly : $$\int_{-\pi}^{\pi} u v'' = -m^2 \int_{-\pi}^{\pi} u(x) v(x) dx.$$

$\boxed{1(d)}$ From (c) & (b), we have

$$-n^2\int_{-\pi}^{\pi} uv = -\int_{-\pi}^{\pi} u'v'\,dx$$

and $$-m^2\int_{-\pi}^{\pi} uv = -\int_{-\pi}^{\pi} u'v'\,dx$$

$$\therefore \quad -n^2\int_{-\pi}^{\pi} uv\,dx = -m^2\int_{-\pi}^{\pi} uv\,dx$$

$$\Rightarrow \boxed{0 = (m^2-n^2)\int_{-\pi}^{\pi} uv\,dx}$$

So if $m \neq n$: then need this to be zero.

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \sin(nx)\cos(mx) = 0 \quad (n \neq m)}$$

Also,

$$\int_{-\pi}^{\pi} \sin(nx)\cos(nx)\,dx = 0$$

since $\sin(nx)\cos(nx)$ is an odd function about $x=0$.

1(e)

$$\sin(nx)\sin(mx) = \frac{-\cos((n+m)x) + \cos((n-m)x)}{2}$$

$$\cos(nx)\cos(mx) = \frac{\cos((n+m)x) + \cos((n-m)x)}{2}$$

so: $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \qquad (n \neq m)$

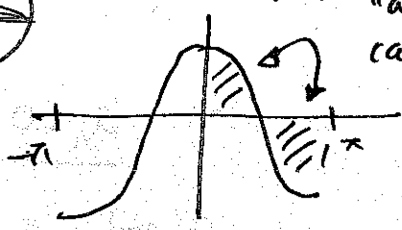
$$= \frac{1}{2}\left\{ \underbrace{\int_{-\pi}^{\pi} \cos((n-m)x)\,dx}_{0} - \underbrace{\int_{-\pi}^{\pi} \cos((n+m)x)\,dx}_{0} \right\}$$

→ since

~~(since cos[(n-m)x] is even)~~ ~~(since cos[(n+m)x] is even)~~

$= 0.$

These two "areas" cancel.



1(f)

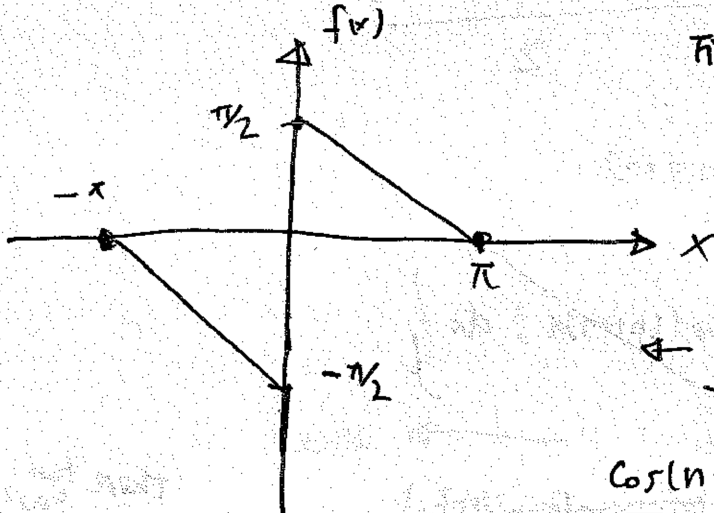
$$\sin^2(nx) + \cos^2(nx) = 1.$$

$$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

so: $\displaystyle\int_{-\pi}^{\pi} 1\,dx = 2\int_{-\pi}^{\pi} \sin^2(nx)\,dx$

$$\Rightarrow \quad \frac{2\pi}{2} = \int_{-\pi}^{\pi} \sin^2(nx)\,dx \longrightarrow \boxed{\pi = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \cos^2(nx)\,dx}$$

$\boxed{10.)}$ $\boxed{1(a)}$ $f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2} & -\pi \le x < 0 \\ \frac{\pi}{2} - \frac{x}{2} & 0 \le x \le \pi \end{cases}$



Find Fourier coefficients : a_n, b_n:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

odd function : i.e. $f(x) = f(-x)$

$\underbrace{\cos(nx)}_{\text{even}}$ $\underbrace{\sin(nx)}_{\text{odd fn.}}$

So: $a_n = 0$ for all n. ($\Rightarrow a_0 = 0$ as well).

So only b_n's are non-zero.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - \frac{x}{2}\right)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left\{ \underbrace{\frac{\pi}{2}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi}} - \frac{1}{2}\underbrace{\int_0^{\pi} x\sin(nx)\,dx} \right\} \quad \ldots \text{eq'n } ①$$

$$\frac{\pi}{2n}\left[\underbrace{-\cos(n\pi)}_{-(-1)^n} + 1\right] = \begin{cases} 0 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd.} \end{cases}$$

$$= \left\{ -\frac{x\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx \right\} = \left\{ -\frac{\pi(-1)^n}{n} + \cancel{\frac{1}{n}\frac{\sin(nx)}{n}\Big|_0^{\pi}} \right\}$$

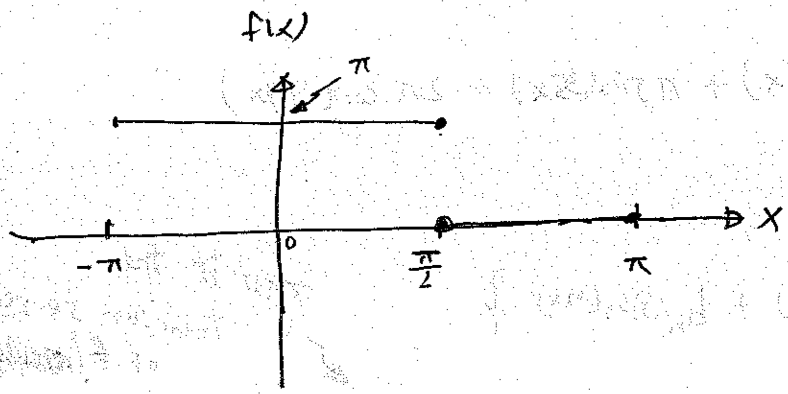
$\cancel{\ldots} \to 0$

$$= \frac{2}{\pi}\left\{ \frac{\pi}{2n}\left[-(-1)^n + 1\right] + \frac{\pi}{2n}(-1)^n \right\}$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2n}\right) = \boxed{\frac{1}{n}}$$

Hence: $$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin(nx)}$$ ← Fourier series for f(x)

1(b) $$f(x) = \begin{cases} \pi, & -\pi \le x \le \pi/2 \\ 0, & \frac{\pi}{2} < x \le \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{\cancel{\pi}}{\cancel{\pi}}\int_{-\pi}^{\pi/2} \cos(nx)\,dx$$

$$= \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi/2}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$= \begin{cases} 0, & \text{if } n \text{ even.} \\ \frac{\pm 1}{n}, & \text{if } n \text{ odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{\cancel{\pi}}{\cancel{\pi}}\int_{-\pi}^{\pi/2} \sin(nx)\,dx$$

$$= \left.\frac{-\cos(nx)}{n}\right|_{-\pi}^{\pi/2}$$

$$= -\frac{1}{n}\left\{ \underbrace{\cos\left(\frac{n\pi}{2}\right)}_{\begin{cases} 0, & n \text{ odd} \\ \pm 1, & n \text{ even} \end{cases}} - \underbrace{\cos(n\pi)}_{\begin{cases} -1, & n \text{ odd} \\ +1, & n \text{ even} \end{cases} = (-1)^n} \right\}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\,dx$$

$$= \pi/2 + \pi = \boxed{\frac{3\pi}{2}}$$

$$a_0/2$$

So: $f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{\sin(n\pi/2)}{n} \cos(nx) - \frac{1}{n}\left[\cos(\frac{n\pi}{2}) - \cos(n\pi)\right] \sin(nx) \right\}$

$$= \boxed{\frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sin\left(\frac{n\pi}{2}\right)\cos(nx) - \left[\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right]\sin(nx) \right\}}.$$

1(c) $f(x) = \underline{53}\sin(2x) + 35\cos(100x) + \pi\sin(35x) + 2\pi\cos(77x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

this is the Fourier series of f!

So: $f(x) = \underbrace{53}_{b_2}\sin(\underbrace{2}_{n=2}x) + \underbrace{35}_{a_{100}}\cos(\underbrace{100}_{n=100}x) + \underbrace{\pi}_{b_{35}}\sin(\underbrace{35}_{n=35}x) + \underbrace{2\pi}_{a_{77}}\cos(\underbrace{77}_{n=77}x)$

All other a_n's and b_n's are zeros.

$\therefore$ $\boxed{b_2 = 53 \quad a_{100} = 35 \quad b_{35} = \pi \quad a_{77} = 2\pi}$ (All else zeroes)

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