MITES 2008 : Physics III - Oscillations and Waves :: Problem Set 1.

Massachusetts Institute of Technology
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(Due on Wednesday, July 2, 2008 before 11:59 PM: Slip under Louis' door in Simmons.)

Problem 1. Simple pendulum.

Derive the equation of motion of a simple pendulum, with a bob of mass m, attached and pivoted at the ceiling by a massless string of length L. In this problem, assume that the angle θ that the bob moves through remains very small (It will help you to remember that when θ is small, $sin(\theta) \approx \theta$ by first order Taylor expansion (you'll learn this in your calculus class eventually). Solve the equation of motion and describe any free parameters (degrees of freedom) in your solution to the equation of motion.

Problem 2. Ions and atoms trapped in an optical lattice.

"Optical lattice" is formed by using a series of interlocking laser beams to create periodic electromagnetic intensity patterns in space. If an ion or a neutral atom sits in one such lattice, it feels an electromagnetic force, whose potential looks like

$$U(x) = U_0 sin^2(kx)$$

Atoms get trapped within the local minima of the potential energy landscape $U(x)$. If many atoms are placed in these local minima (think of eggs sitting in an egg carton), they form a crystal in that the atoms are periodically spaced. Optical lattice is a promising tool for as yet unrealized dream of building quantum computers. Early on, the major contributers to this area of research were three physicists with MIT connections (Prof. Eric Cornell (MIT physics Ph.D. 1990), Prof. Carl Wieman (MIT physics S.B. 1973), and Prof. Wolfgang Ketterle (current MIT physics faculty)) who were awarded the 2001 Nobel prize in physics for creation of a long-sought after quantum material called "Bose-Einstein Condensate" using optical trapping of the type discussed here. In this problem, we investigate the motion of small oscillation of an atom about the minima of the optical lattice potential.

(a). Plot the potential energy landscape $U(x)$. What is the physical meaning of k? What is the physical meaning of U_0?

(b). Assume that an atom of mass m is resting in one of the minima of $U(x)$. For definiteness, let's say it's sitting at a potential minimum at $x = 0$. If I perturb it by a very small distance δx, what is the resulting motion of the atom? To answer this, derive the equation of motion of the atom, and solve it. What do we mean by small distance δx? Small compared to what? It may help you to remember that $sin(z) \approx z$ if $z \ll 1$. Can you justify this approximation graphically?

Problem 3. Sand filling bucket-oscillator

A bucket of mass M is initially empty. Starting at time $t = 0$, sand starts to be poured directly into the bucket at some unknown rate. This bucket is also attached to the wall through a Hookian spring with stiffness (spring constant) k. Assume that the floor on which the bucket would slide on is frictionless. Let's denote m(t) to be the mass of the bucket plus the mass of sand collected in the bucket at time t. We want to understand how the bucket moves in this problem.

(a). What is wrong with the following logic?
"The only difference between this problem and the simple harmonic block-spring oscillator we studied in class is that the mass of the block (bucket+sand) is changing in this problem. But this is easy to handle: we just have to say m is no longer a constant but is a function of time, $m(t)$. In other words, the equation of motion is no longer

$$m\frac{d^2x}{dt^2} + kx = 0, \tag{1}$$

where m is treated as a constant, but is now the following:

$$m(t)\frac{d^2x}{dt^2} + kx = 0. \tag{2}$$

Hence, the angular frequency is just $\omega(t) = \sqrt{\frac{k}{m(t)}}$, *which is now a function of time."*

(b). Now, let's find out the motion of the block the proper way. First, start with Newton's second law. Then work through the algebra, leaving the rate at which sand is gained as $\frac{dm}{dt}$ in your equations. What is the equation of motion? Give a physical interpretation of various terms you see in your equation of motion.

(c). Write down the expression for the total energy of the system.

(d). The total energy $E(t)$ is a function of time. What is the rate of change of this energy? Your final answer should be a singe term involving only $\frac{dm}{dt}$ and $\frac{dx}{dt}$, with a number in front, but no other terms. Looking at this term, say if $E(t)$ is increasing or decreasing over time. Since energy $E(t)$ is changing over time, is the conservation of energy being violated? Where is the energy going to? (i.e. why does the energy change?)

(e). Solve the equation of motion if the total mass of bucket plus sand as a function of time t is described by $m(t) = M + \beta t$. Comment on the behavior of the bucket + sand system for some representative values of β. In this problem, we assume that we're restricting ourselves to time range short enough that $M \gg \beta T$, where T is the longest time interval that we'll allow ourselves to observe the motion of block. (i.e. This will allow us to roughly estimate $\omega_0 \approx \sqrt{\dfrac{k}{M}}$.

Problem 4. Oscillating through a tunnel in Earth (aka. Tunnel vs. orbit)

Which one of the following two methods will get you faster from the North pole to the South pole of Earth: Using a straight tunnel dug through the Earth connecting the two poles, or by riding a satellite in a uniform circular orbit from North pole to South pole? Both use Earth's gravitational field for navigation, but which one is faster? In this problem, we study the motion of a particle (person) through the tunnel, which turns out to be oscillatory between pole to pole.

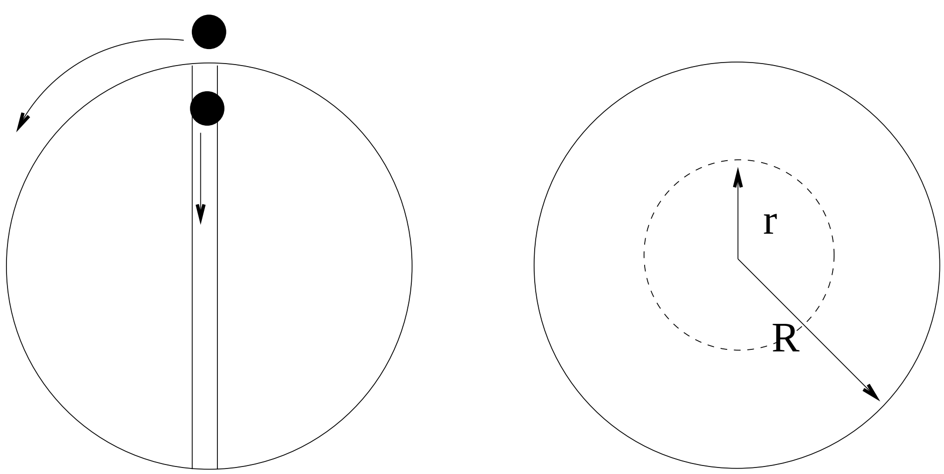

FIG. 1: Tunnel through the Earth or orbit around the Earth?

(a). By deriving the equation of motion, show that a person oscillates between the North and the South pole (i.e. show that the Eqn. of motion is that of simple harmonic oscillator). For this purpose, we need to calculate the gravitational force on a person falling through the tunnel. Let's do this step-by-step as follows:

(i). What is the total mass of the Earth that is enclosed within the dashed sphere of radius r (See Fig. 1). Let M be the total mass of the Earth and assume that the Earth is a perfect sphere of radius R.

(ii). What is the force of gravity at radial position $\vec{r}$, due to only the mass enclosed within the dashed sphere? (Remember, force has both a direction and a magnitude: state both). Let G be the Newton's gravitational constant.
(iii). It is more tricky to calculate the gravitational force at position $\vec{r}$ due to the mass outside the dashed sphere. Nevertheless, *without* doing *any* calculation, we can use the concept of **symmetry** to figure out which directions the force *cannot* point in. Using diagrams and the idea of symmetry (and no calculations), explain which directions the force at position $\vec{r}$ cannot point to and draw the *two* directions that it *may* point to, due to the gravitational force of the spherical slab lying outside the dashed sphere in Fig. 1.

(iv). It turns out that the force on the person at position $\vec{r}$ is due only to the mass enclosed within the dashed sphere; the gravitational force at position $\vec{r}$ due to any part of Earth that's outside the dashed sphere is zero. To prove this, show that the gravitational force inside a *spherical shell* is zero by showing that the pieces of mass at the ends of the cones in Fig. 2 give canceling forces at point P. Then, use this to show that indeed the gravitational force at position $\vec{r}$ due to mass of Earth outside the dashed sphere is zero.

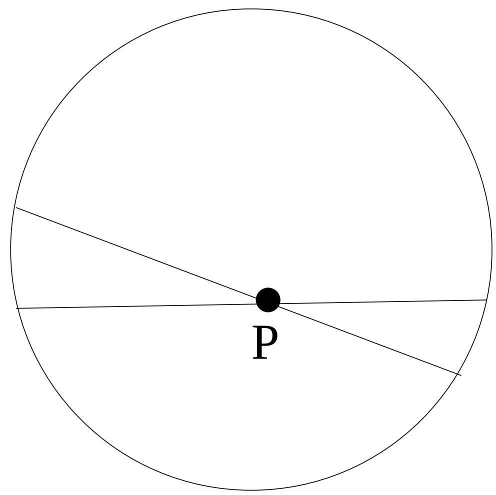

FIG. 2: Spherical shell (i.e. Hollow core, all the mass is slapped on the surface of the sphere (i.e. empty watermelon)

(v). Now that you know the force acting on the person at $\vec{r}$, write down the equation of motion of the person, and show that she/he is oscillating back and forth between the two poles of Earth. What is the angular frequency ω_0 of this oscillation? Does he/she oscillate faster (higher frequency) or slower (lower frequency) if the mass of the Earth were to increase? What would happen to the oscillation frequency if the mass of the *person* were to increase? Also, what would happen to the oscillation frequency if the radius of the Earth R were to increase?

(b). From the frequency obtained in (v), determine the time taken for the person to go from North pole to the South pole (i.e. After being "released" at the North pole, the person would travel through the tunnel and reach the South pole. Just before being turned back (due to gravity), the person jumps off the tunnel at the South pole. What is the time taken for this trip)?

(c). What is the speed required for a uniform circular orbit of radius R about the Earth? (i.e. a circular orbit at constant speed v that just grazes over the Earth's surface). Does this speed increase or decrease if the Earth's radius were to increase? (In this problem, we're ignoring the orbit of the Earth itself, and also the spin of the Earth as well. Earth is just a solid sphere of radius R that remains stationary).

(d). Starting from the North pole, how much time does a spacecraft in the circular orbit in (c) take to get to the South pole?

(e). Finally, we're ready to answer the question posed at the very beginning of this problem. Which of the two methods gets you faster to the South pole from the North pole: The tunnel or the orbit? And for fun, which mode is less dangerous for a human being?

Problem 5. Beads sliding on a wedged wire frame.

Two beads of mass m are constrained to slide along the wedged wire frame shown in Fig 3. The wedge spans angle 2θ. These two beads are attached to each other by a spring with Hooke's spring constant k. The acceleration due to gravity is g. The wire frame is well-oiled, and thus assumed to be frictionless.

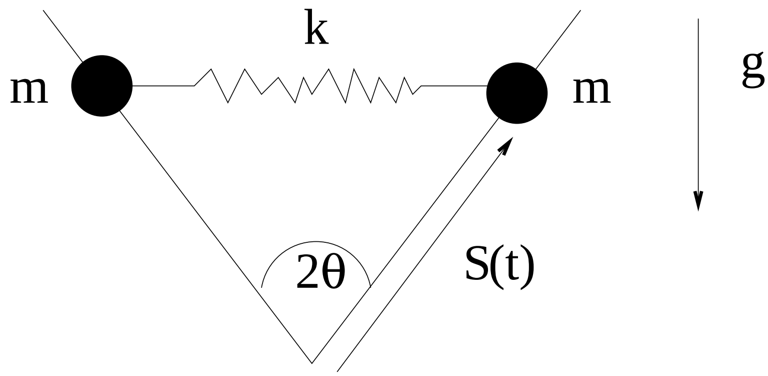

FIG. 3: Wedged wire frame.

(a). What is the equilibrium position of the two beads in this system? Write your answer in terms of distance s_0 from the vertex of the wedge along the wire frame.

(b). If you were to perturb the two beads about their equilibrium position (by, for example, sliding both beads up by a same amount along the wire frame), what is the ensuing motion? Show that a simple harmonic oscillation results by deriving their equation of motion for the case in which the spring remains parallel to its equilibrium position. What is the frequency of their oscillation? Solve the equation of motion and find their position along the rail as a function of time.

Problem 6. Attaching springs together in series.

(a). Using *only* Newton's 2nd law, show that when two *massless* springs of spring constant k_1 and k_2 respectively, are joined together end-to-end, the resulting spring constant is $\dfrac{k_1 k_2}{k_1 + k_2}$. You are NOT allowed to use any "resistor" analogy!

(b). What would the effective spring constant be, if you had N identical springs (each with spring constant k) attached to each other, one after another in a series? Again, use *only* Newton's 2nd law to derive your result.

Problem 7. Massive spring.

Real springs, made up of atoms, must have a mass themselves. In this problem, we investigate how the angular frequency ω changes for an oscillation of a block that is attached to a massive spring.

A block of mass M is suspended at the end of a spring with a spring constant k. We assume that the mass of the spring is m. Suppose that in equilibrium, the spring has length L. Imagine an infinitesimal element of spring with length dy which is located at distance y below the fixed end of the spring at the ceiling.

(a). Show that the kinetic energy of this infinitesimal element of spring is

$$\frac{1}{2}\frac{m}{L}dy\left(\frac{yv}{L}\right)^2$$

where v is the velocity of the suspended block.

(b) Show that the total kinetic energy of the spring is $\frac{mv^2}{6}$. Finally show that the angular frequency ω of oscillation is

$$\omega = \sqrt{\frac{k}{M + m/3}} \tag{3}$$

Problem 8. Probability of localizing a simple harmonically oscillating particle.

A particle is oscillating simple harmonically in one dimension (call it the x-axis). It has amplitude A and takes infinitesimal time dt to move from position x to $x + dx$. Show that the probability of localizing (finding) the particle between x and $x + dx$ is

$$\frac{dx}{\pi(a^2 - x^2)^{1/2}}.$$

It is interesting to note that if the particle is quantum mechanical (i.e. a quantum mechanical simple harmonic oscillator such as a single atom trapped in a quadratic potential well), the probability of finding the particle outside its "classically allowed" region ($-A < x < A$) is NOT zero. This is due to the wave nature of a particle which allows for it to "seep through" the potential well.

Problem 9. Damped oscillator summary.

Explain in your *own* words and equations what a damped simple harmonic motion is. Be sure to explain what a (i) under damped, (ii) critically damped, and (iii) over damped oscillation regimes are.

Problem 10. A dangerous gamble...

Using your knowledge of complex numbers that we discussed in class, answer the following question:

If I offered you i^i dollars, in exchange for your one dollar, would you accept my offer?

Be sure to justify your answer.