NB1140: Physics 1A - Classical mechanics and Thermodynamics
Problem set 1 - Describing motion of objects (Kinematics)
Week 1: 14- 18 November 2016

Problem 1. Squished between two trains (Zeno's paradox)
In this problem, you will show that you can take an infinite number of steps and yet walk only a finite distance.

Two trains are heading towards each other (Figure 1). Both trains are moving at a constant speed V, with one train moving to the right and the other moving to the left. Initially (at $t = 0$), the two trains are separated by a distance D. A fly is trapped between the two trains. The fly flies at a constant speed u, first going from the left to the right train, and then flying back to the left train, and then back to the right train, and so forth. The fly does this until it gets smashed by the two colliding trains. Here we assume that the fly is a point object (i.e. it takes up zero volume). We also assume that when the fly reaches one of the trains, it immediately turns around (i.e. it takes zero time to turn around and start flying back in the other direction at speed u). Also assume that $u > V$ (yes, here we have a superfly that moves faster than the trains).

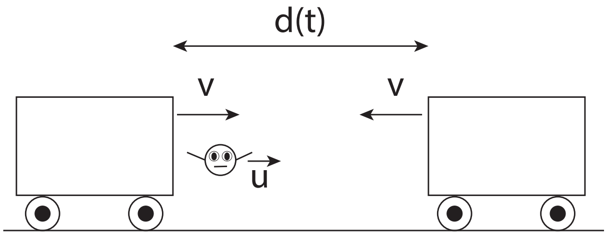

Figure 1: Two trains squishing a fly

(a) Let $d(t)$ be the distance between the two trains' heads at time t. Determine $d(t)$ as a function of D, and V.

(b) At what time t_f do the two trains collide with each other? Express your answer in terms of D, and V.

(c) What distance does *each* train travel before they collide? (i.e. distance travelled between $t = 0$ and $t = t_f$)?

Note that above answers don't depend on u because how the trains move is independent of what the fly is doing.

(d) In (c), you calculated the answer. Now, without any math, explain in words why the answer *must* be what you calculated in (c). In other words, why can't the answer be that the trains meet at a location closer to one side than the other? [*Hint: What happens if you look at Figure 1 after "flipping" the page (or look at it in the mirror)? Does Figure 1 look the same to you or not?*]. This explains why your ansewr in (c) does *not* depend on V.
The answer you got in (c) might seem obvious. If so, it's because the notion of **symmetry** is ingrained in your mind from your (unconcious) everyday experience. In many situations in physics, the notion of symmetry like this one can help you get the answer without doing

lots of calculations. In this particular problem, you didn't have to do much calculations but later in the course, you'll see examples where you can avoid doing a whole page of calculations by invoking the concept of symmetry. You don't always have to calculate your answer in physics (and on exams). Invoking physical argument to get the answer in just 1-2 lines is just as valid (and can give you full points, if your reasoning is justified properly).

(e) What is the total distance travelled by the fly before it is squished? Does this answer depend on its initial position between the trains at $t = 0$? (i.e. Does your answer depend on whether the fly started from the left train or the right train or 1/4 of the way between the two trains, etc? Why or why not?)

Let's calculate the same answer as in (e) but with a different method. Assume that initially (at $t = 0$), the fly starts at the head of the left train. It starts flying at speed u. Both trains are still moving at the constant speed V towards each other. We again assume that $u > V$ and that the fly takes zero time to turn around and fly back in the other direction (so it's speed is always u).

(f) What is the time interval Δt_1 taken by the fly to go from the left train to the front of the right train? Also, what is the distance d_1 that the fly flies during this time Δt_1? Express your answer in terms of D, V, and u.

(g) Now, after arriving at the front of the right train, the fly flies back to the front of the left train. We want to calculate the time interval Δt_2 taken and the distance d_2 covered by the fly during this second trip (going from right train to left train). Let's do this step by step:

- First, let D_2 by the distance between the left train and the fly after time interval Δt_1. By drawing a picture, show that $D_2 = (u - V)\Delta t_1$ (Remember, $u > V$ so the left train is moving too slow to catch up with the fly during the first one-way trip in part (f)). One way to see this result is using "relative motion". As a passenger on the left train, the fly's speed relative to you (and thus the left train) is $u - V$. And relative to you, the left train is not (and you are not) moving. So the fly is getting aways from you and the left train during the first one-way trip at a speed $u - V$ for time interval Δt_1. Thus the distance between the left train (you) and the fly must be $(u - V)\Delta t_1$ at the end of the first one-way trip.

- Now, relate D_2 with Δt_2 by writing D_2 on one side of an equation and Δt_2 multiplied by some factor on the other side of the equation (like you related D with Δt_1 in part (f). In part (f), we can consider $D_1 = D$).

- Using above two equations, $D_2 = (u - V)\Delta t_1$ and the other you found by relating D_2 with Δt_2, calculate Δt_2 in terms of Δt_1. [*Answer*: $\Delta t_2 = \frac{u-V}{u+V}\Delta t_1$]

- Finally, calculate d_2. [*Answer*: $d_2 = u\frac{u-V}{u+V}\frac{D}{u+V}$]

(h) Let's generalize the results of (f) and (g). We want to calculate the time interval Δt_n taken and the distance d_n flown by the fly during the n-th one way trip. We will do this step-by-step like in part (g). Actually, what we will do is **derivation by induction** (which is like the **proof by induction** that you learned in your math courses).

- Let n be some large integer. Let D_n be the distance between the "inbound train" (the train that's moving towards the fly) and the fly at the beginning of this n-th one-way flight. This one way flight happens during time interval Δt_n. At the end of this one-way flight, the inbound train and the fly meet. By drawing a picture, show that $D_n = (u - V)\Delta t_{n-1}$. You can also use the concept of relative motion to derive this result as in part (g).

- Now, relate D_n with Δt_n by writing D_n on one side of an equation and Δt_n multiplied by some factor on the other side of the equation.

- Using above two equations, calculate Δt_n in terms of Δt_{n-1}. [*Answer*: $\Delta t_n = \frac{u-V}{u+V}\Delta t_{n-1}$]

- Calculate d_n in terms of t_{n-1}. [*Answer*: $d_n = u\frac{u-V}{u+V}t_{n-1}$]

- Let's now use induction to write d_n in terms of just u, V, and D. Since we have

$$t_n = \frac{u-V}{u+V}t_{n-1}$$
$$d_n = u\frac{u-V}{u+V}t_{n-1} \tag{1a}$$

  show that

$$d_n = u\left\{\frac{u-V}{u+V}\right\}^{n-1}\frac{D}{u+V} \tag{2}$$

  Note that above equation is true even for *all* values of n, not just for large n. For example, for $n = 1$ and $n = 2$, you see that you get the d_1 and d_2 that you got in parts (f) and (g).

(i) Now, compute the total distance flown by the fly by computing the following infinite sum

$$d_{total} = \sum_{n=1}^{\infty} d_n \tag{3}$$

Here, you can use the formula for doing an *infinite geometric sum* that you learned in your analysis course or derive it yourself. Why do you compute a sum of infinite number of d_n's instead of a finite number of them? Note that you should get the same answer as in part (e).

Here you have just shown that you can take an infinite number of steps (flights), and yet cover a finite distance in a finite amount of time. This is called a Zeno's paradox.

(j) Now for the ultimate challenge: Would this whole situation work for a real fly? [Answer: No! Zeno's paradox does not apply to a real fly]. Why not? [Hint: A real fly is not a point object because it has a non-zero volume. Now think of how a non-zero volume affects the answer you calculated above. Just to be concrete, say the fly is a sphere with a diameter h. For what value of n does d_n start to become smaller than h?]

***Problem 2: Why bugs spiral into light (logarithmic spiral)**

Parts (e) and (f) of this problem will appear on Quiz 1

One way to kill an annoying bug that flies is by putting a hot lamp that emits a bright light. The bug flies into the lamp. Once the insect lands on the lamp, it is cooked to death by the immense heat. But you will often see that the fly does not fly straight into the lamp. Often, it spirals into the lamp. Why does this happen? In this problem, we calculate the exact trajectory of the bug.

When there is a bright light (like a full moon in a clear sky at night), many birds and insects like to keep a specific angle between their eyes and the light source. This helps them navigate the Earth. The reason that birds and bugs don't spiral into the moon is that the moon is too far away. That's not the case if we put a bright lamp in the same room as the bug! So we're really hijacking a naturally evolved navigational mechanism to kill the bugs. Let's calculate how this actually happens.

Suppose we have a bug (see the picture on the right). Initially, it is at a distance R from the lamp. The bug always flies at a constant speed V but **not** at a constant velocity. This is because the bug always maintains an angle θ between its direction of flight and the lamp. This means that the velocity vector $\vec{V}$ is always at a constant angle θ with respect to the radial line that joins the bug and the lamp. (Note: $|\vec{V}| = V$). We can always decompose the velocity vector $\vec{V}$ into two component vectors that are perpendicular to each other: $\vec{V_{||}}$ and $\vec{V_{\perp}}$ (see picture). We can do this at all times, during the bug's entire flight. $\vec{V_{||}}$ is always parallel and aligned with the radial line that joins the lamp and the bug. The $\vec{V_{\perp}}$ is always perpendicular to this radial line.

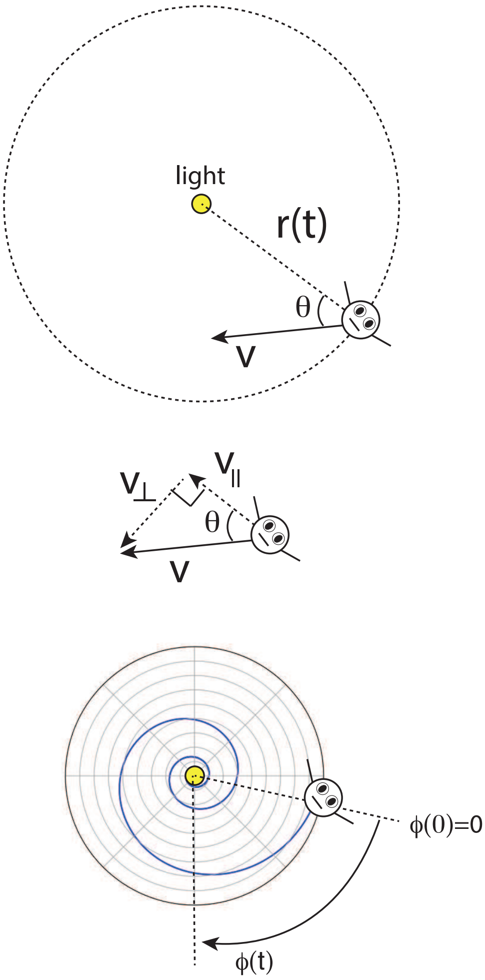

Figure 2: A bug spirals into a lamp. Bottom figure is from Wikipedia.

Let's do a simpler example first. Suppose $\theta = \frac{\pi}{2}$ (in radians. Remember that in physics, we measure angles in radians).

(a) What are the lengths of the vectors $\vec{V_{\perp}}$ and $\vec{V_{||}}$ in terms of V?

(b) Describe the bug's motion in words. Will it ever hit the lamp?

Let's now assume that $\theta = 0$.

(c) What are the lengths of the vectors $\vec{V_{\perp}}$ and $\vec{V_{||}}$ in terms of V?

(d) Describe the bug's motion in words and then mathematically derive its motion by calculating $r(t)$ (with r(0) = R). When does the bug land on the lamp? [Answer: R/V].

Now, let's analyze how the bug spirals into the lamp. Suppose $0 < \theta < \frac{\pi}{2}$.

(e) What are the lengths of the vectors $\vec{V_\perp}$ and $\vec{V_{||}}$ in terms of V and θ?

(f) Calculate $r(t)$ in terms of V, θ, and R. When does the bug arrive at the lamp? *Answer:* $r(t) = R - Vtcos(\theta)$.

(g) Calculate $\phi(t)$ in radians (see bottom of Figure 2). We will study angular motion in detail later in this course. But for now, you can use the fact that at given instant of time t, the *instantaneous angular velocity* $d\phi/dt$ is given by

$$r(t)\frac{d\phi}{dt} = V_\perp \tag{4}$$

By solving above differential equation, show that

$$\phi(t) = A \cdot log(B(t)) \tag{5}$$

Here log is the natural logarithm (base e). Here you have to show what A and $B(t)$ are in terms of θ, t, V, and R.
[Hint: Remember from your analysis courses that $\int \frac{1}{(1+x)}dx = log(1+x) + constant$]

The resulting trajectory of the bug is defined by both $r(t)$ and $\theta(t)$. This path is the blue spiral shown at the bottom of figure 2. This path is called the logarithmic spiral because of the log(B(t)) in $\phi(t)$. We can see this shape in many different objects in nature: logarithmic spirals of galaxy, cyclones, broccoli, and sea shells, just to name a few.

Problem 3: Travelling faster than light

You probably heard that nothing can travel faster than light (except for light itself of course). Indeed, any object with a mass cannot travel faster than light. This is a consequence of Einstein's theory of relativity. Even any massless particle such a photon (particle of light) cannot travel any faster than the speed of light. Well, that's actually not quite true. In this problem, we show that there's a caveat: Some things *can* travel faster than light! But no signal (information or energy) can travel from point A to point B faster than light. This is important because it means that you *cannot* have an effect *before* a cause (e.g., getting shot by a bullet before the gun is fired, or being born before your mother was born, etc.).

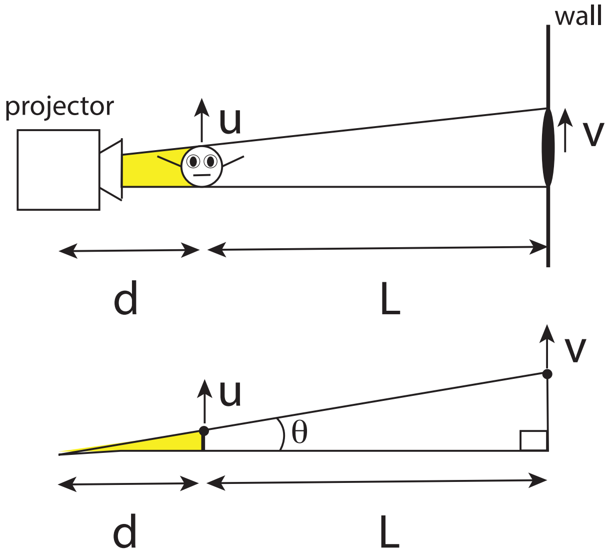

Figure 3: Shadow of a fly on a wall.

Consider the set up in the picture to the right. A fly is in front of a projector that emits light. The fly is flying upwards at a constant speed u. It is blocking the light from the projector. So the fly casts a shadow on the wall behind. The shadow moves up the wall at a speed v due to the fly moving up. The fly is at a distance d from the projector and is at a distance L from the wall. We assume that $d << L$ (i.e. d is much smaller than L). To model this situation in a very simple way, let's consider the right-angled triangle in the bottom of the figure. This triangle consists of two right angled triangles: the yellow triangle and the bigger triangle (which includes the yellow one). Both triangles share the same pointy wedge. This wedge has an angle θ. When the fly moves up at the speed u, we can consider one side of the yellow triangle to increase in its length at speed u. Similarly, we can consider one side of the bigger triangle to increase in its length at speed v.

(a) Calculate d(tanθ)/dt in terms of d and u.
[*Hint: Calculate the rate (length / time) at which the yellow triangle's side length changes over time*].

(b) Using your result from (a), perform step-by-step calculations to prove that

$$v = (d + L)\frac{u}{d} \tag{6}$$

(c) Let c be the speed of light. By rearranging the terms in above equation, show that for the shadow to travel at a speed faster than the speed of light ($v > c$), we need

$$\frac{L}{d} > \frac{c}{u} - 1 \tag{7}$$

(d) Looking at above equation, note that by making the fly as close to the project as possible, we can make d as close to zero as we want. In this way, we can satisfy the inequality (i.e. make the left hand side $\frac{L}{d}$ larger than the right hand side $\frac{c}{u}$ - 1). Thus we can indeed arrange the fly to be sufficiently close to the projector so that the fly's shadow on the wall travels faster than light. To get a feel for how far the fly should be to the projector and how far the wall should be, let's put in some numbers. In fact, instead of a fly, suppose we have a mini Boeing 747. A Boeing 747 can travel at a speed of approximately 1000 km/h relative to the ground (called the *ground speed*). Speed of light is 3x10^8 m/s. Suppose the mini 747 is 1 mm away from the projector. How far should the wall be from the 747 so that the airplane's shadow travels at the speed of light? [Answer: You should get a number that is close to about 100 m].

Problem 4. Bacterial chemotaxis: Random walk
In lecture note 1, you learned how bacteria like E. coli swim towards food. Let's analyze a highly simplified model of this behaviour. Let's assume that an E. coli moves along a line. So it can only move to the left or right along this line.

(a) Suppose the E. coli's receptors for detecting the chemoattractants is defective. This can happen if there is a mutation in the DNA sequences that code for the genes that make up the receptor. In this case, even when there is food, the E. coli cannot smell it (because it has no or defective receptors on its cell membrane). As a result, the E. coli cell swims to the right at a constant speed V for a fixed time interval T, and then it immediately turns around, and then swims to the left at a constant speed V for a fixed time interval T. And then it repeats this motion over and over (first to the right, and then to the left again). Sketch the graphs of the position and velocity of the E.coli as a function of time t. Watching over a long time, what is the average displacement of the E. coli relative to its initial position as a function of time?

(b) Suppose the E. coli now has working receptors. The cell makes not just one but many copies of this receptor and displays them on its outer cell membrane. Some of the receptors will have the *chemoattractant* (attractive food molecules) bound to them. How many receptors are bound to the food molecules depends on how many food molecules there are (In your chemistry course, you will probably learn how to calculate this quantity by assuming a **chemical equilibrium**: the answer depends on the concentration of the food molecules, the concentration of receptors on the cell surface, and an "equilibrium constant"). When the receptors are bound to food molecules, they trigger intracellular signalling events that cause the cell to spend more time swimming in the direction of the food than in the other direction. This allows the cell to swim up the concentration gradient (it goes from lower to higher concentration of the food molecules). Let's model this situation. The food (sugar) is at the right side of the E. coli's initial position. Smelling food, it swims to the right at a constant speed V for a fixed time interval T_R, and then it swims to the left at a constant speed V for a fixed time interval T_L. Importantly, the intracellular signalling events ensure that $T_R > T_L$.

Sketch the graphs of the position and velocity of the E. coli as a function of time t. Watching over a long time, what is the average displacement of the E. coli relative to its initial position as a function of time?