NB1140: Physics 1A - Classical mechanics and Thermodynamics
Problem set 2 - Forces and energy
Week 2: 21- 25 November 2016

Problem 1. Why force is transmitted uniformly through a massless string, a massless spring, and any massless objects

(**a**) Consider an object of mass m. Write down the Newton's 2nd law. Then by saying that $m = 0$ (i.e. the block is *massless*), show that the total force (also called "net force") on the object must be zero.

(**b**) Suppose that there is a non-zero net force on the massless object. By writing $\vec{F}_{tot}/m$, show that the acceleration of the object must be infinite. This is clearly impossible. No object can have an infinite acceleration (or infinite speed). So indeed the total force acting on any massless object must be zero.

(**c**) Consider a massless string. Let's analyze a short segment of the string (black segment in Fig. 1A). This short segment has zero mass (since the entire string is massless). It is pulled to the right by the part of the string to its right. It is also pulled to the left by the part of the string to its left. Draw a force diagram that represents all the forces acting on this piece. How is the pulling force from the right side (T) related to the pulling force from the left side (T_L)? [Answer: $T_L = T$].

(**d**) Note that your result in (c) holds for all short (or long) segments of the string. Each segment of the string is pulled equally to the right as it is to the left. This is why we call it a "tension": Each segment of the string is stretched due to two opposing forces (of magnitude T) pulling in opposite directions. Now consider a block attached to one end of the string and your hand pulling the other end of the string

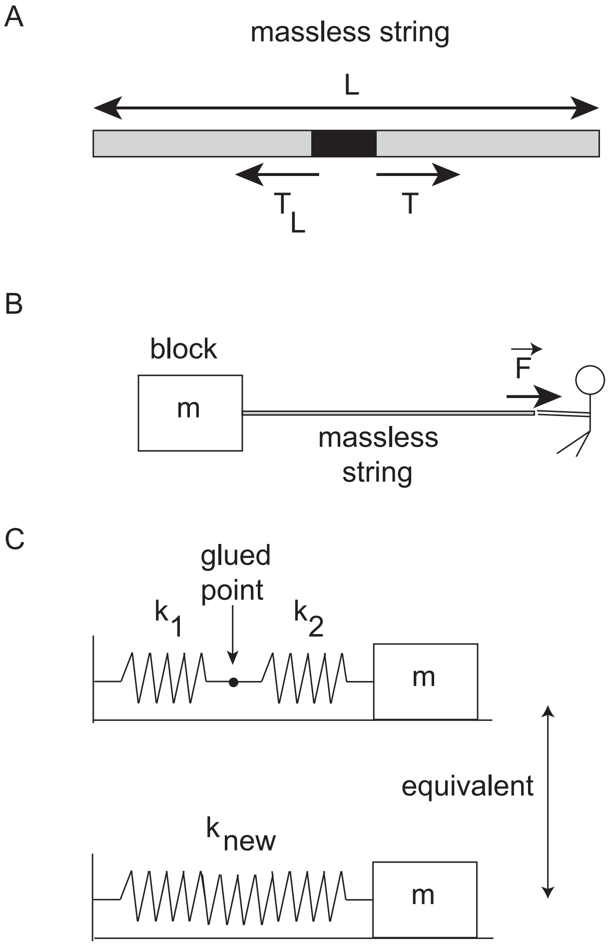

Figure 1: (**A**) Massless string of length L. Black segment is a tiny segment of the string that is pulled to the right with a force of magnitude T and to the left with a force of magnitude T_L. (**B**) Block of mass m is attached to a massless string. A person pulls the right end of the string with a force $\vec{F}$. (**C**) Upper panel: Two massless Hookian springs with spring constants k_1 and k_2 are joined together. The junction ("glued point") is massless like the springs. Bottom panel: Joining two springs gives us a single new Hookian spring with a spring constant k_{new}.

with force $\vec{F}$ (Fig. 1B). By breaking the string into little pieces as in (c), show that the force that the string pulls on the block must also be $\vec{F}$. Draw a free body (force) diagram to justify your answer.

Another way to answer this: Consider your hand, the string, and the block as individual objects. You exert force $\vec{F}$ on the string, and the string exerts reaction force back on your hand, and the string also exerts force on the block. By treating the entire string as a massless single object (instead of breaking it into pieces), show that the string must exert a force $\vec{F}$ on the block. Draw a free body (force) diagram to justify your answer.

(**e**) Now consider two massless springs that follow Hooke's law for springs (Fig. 1C). These are called "Hookian springs". One spring has spring constant k_1 and the other has a spring constant k_2. They are joined together at a single point, which is also a massless point like the rest of the spring. The spring with spring constant k_1 has its left end attached to a wall and its right end glued to the other spring. The spring with the spring constant k_2 has its other end attached to a block of mass m. Now suppose that the left spring is stretched by a distance Δx_1 and that the right spring is stretched by a distance Δx_2. Draw a free body (force diagram) that shows all the forces acting on this massless "glued point" where the two springs are joined (Fig. 1C).

(**f**) Given the same set up as in (e), express Δx_2 in terms of Δx_1, k_1, and k_2. [*Hint*: The point is *massless* (see part (b))].

(**g**) Given the same set up as in (e), what is the force acting on the block of mass m? Assume there is no friction.
[*Hint*: The block is directly attached to only one of the springs. So it doesn't know anything about the spring with spring constant k_1].

(**h**) By joining the two springs together, we create a single new Hookian spring whose spring constant is k_{new} (i.e. when you look at the system, you cannot actually tell that two springs are joined together. Instead, you just see one spring). In this view, we would interpret the situation in (e) in the following way: You pull the block to the right and as a result, you stretch this single spring by a distance Δx, where $\Delta x = \Delta x_1 + \Delta x_2$. The force that the resulting new spring exerts on the block is $k_{new}\Delta x$ to the left (since the spring is stretched to the right). Using your results in (f), and (g), express Δx in terms of Δx_1, k_1, and k_2.

(**i**) Using (f), (g), and (h), express k_{new} in terms of k_1 and k_2.
[*Answer*: $k_{new} = (k_1 k_2)/(k_1 + k_2)$]

(**j**) Suppose you have a spring that's neither stretched nor compressed (i.e. the spring is at its "rest length"). It has a spring constant k. You cut the spring in half. As a result, you get two identical springs that are half the length of the original spring. Using your result in (i), calculate the spring constant of each "half" spring.
[*Answer*: $k_{half} = 2k$]

Problem 2. Three blocks

Consider three blocks of masses m_1, m_2, and m_3. Block of mass m_1 is joined to a block of mass m_2 by a massless rope, and the block of mass m_2 is joined to the block of mass m_3 by a massless rope. The force of gravity is constant in this problem. The acceleration due to gravity is $\vec{g}$ (a vector that points downwards). Let $g = |\vec{g}|$ (a positive number that is the magnitude of the acceleration due to gravity). The rope that connects m_2 with m_3 is wrapped around a massless and frictionless pulley wheel (Fig. 2).

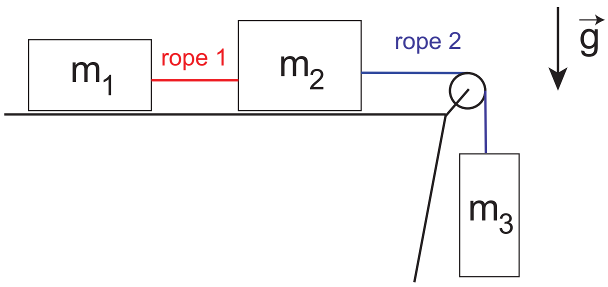

Figure 2: Three blocks with a massless and frictionless pulley

In questions (a $\sim$ c), assume that there is no friction anywhere.

(**a**) For each block, draw a diagram that shows all the forces acting on that block.

(**b**) For each block, what is the acceleration? Give both the magnitude and the direction of acceleration for each block.

(**c**) Calculate the tension T_1 in the rope that joins m_1 to m_2. Also, calculate the tension T_2 in the rope that joins m_2 to m_3.

(**d**) Now suppose that there is friction between the floor and the two masses on top of it. The coefficient of static friction bewteen the table and each block on the table is μ_s. All blocks are initially at rest and are held together by your hands. For what values of μ_s will the system of blocks start to move after you release your hands?

***Problem 3. Optical tweezers - Forces, work done, and energy**
Parts (b) and (c) of this problem will appear on Quiz 2

As we discussed in the lecture, an optical tweezer is a device that consists of a laser light and a micro-bead (e.g. spherical polystyrene bead of diameter $\approx 1\ \mu m$) (Fig. 3A). The bead is trapped at the center of the beam (call it $x = 0$). The optical tweezer exerts a spring-like restoring force on the polystyrene bead that tends to bring back the bead to the beam center if the bead is displaced from the center. The light does this by constantly bombarding the bead with particles of light called "photons". This bombardment tends to bring back the bead to the center of the trap ($x = 0$). Importantly, the laser light exerts this restoring force that is proportional to the displacement from the center. Thus the laser light beam behaves like a Hookian spring. Suppose in the set up, the bead is attached to a DNA. The other end of the DNA is attached to a glass slide (Fig. 3A). The DNA can be stretched and pulled, like a spring.

We can model the optical trap setup as a block of mass m attached to two opposing Hookian springs (Fig. 3B). One spring, representing the DNA, has a spring constant k_{DNA}. The other spring, representing the laser beam, has a spring constant k_{trap}. Let $x = 0$ be the trap center (where the optical tweezer does not exert any force). The trap center $x = 0$ is so far away from the glass slide that the DNA is actually stretched by distance L when the bead is at $x = 0$.

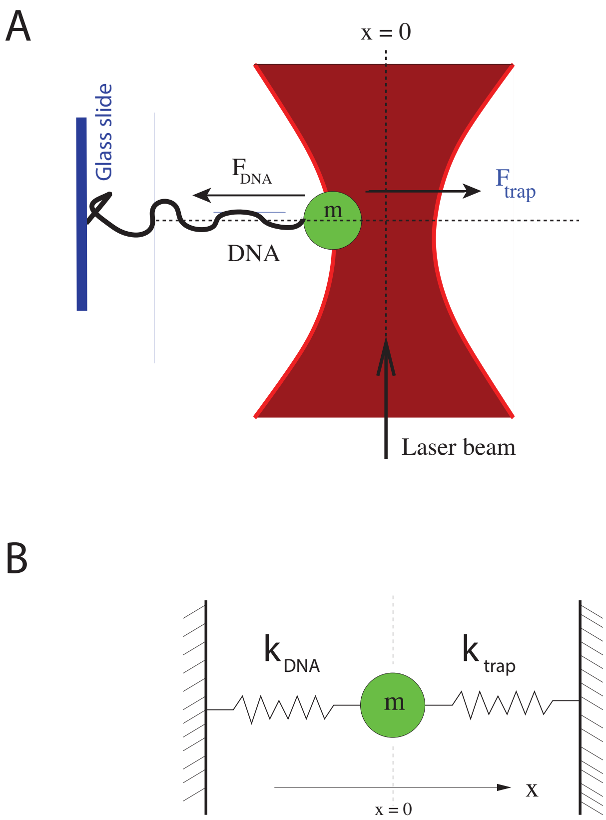

Figure 3: (**A**) Optical trap set up. One end of DNA is attached to a glass slide and the other end is attached to a polystyrene bead (green circle) of mass m. It is trapped by a beam of laser light (red) whose shape and intensity is precisely sculpted by lenses. (**B**) We can model the optical trap set up as a block (representing the bead) of mass m that is attached to two opposing springs, one with a spring constant k_{DNA} (representing spring-like DNA) and another one with a spring constant k_{trap} (representing spring-like optical trap).

(**a**) Write down the Newton's 2nd law equation ($\vec{F}_{net} = m\vec{a}$) for the bead.

(**b**) The equilibrium position is the position of the bead at which the bead has zero net (total) force acting on it. When the bead is at the equilibrium position, by how much is the DNA stretched? [*Answer*: $L(1 - \frac{k_{DNA}}{k_{DNA}+k_{trap}})$]

(**c**) What is the total energy (kinetic energy + potential energy) of the system when the block is at equilibrium position and not moving? Here, you can leave your answer in terms of k_{DNA}, L, k_{trap}, and k, where we define $k = k_{DNA}/(k_{trap} + k_{DNA})$.
[*Answer*: $\frac{L^2}{2}(k_{DNA}(1-k)^2 + k_{trap}k^2)$]

Now, when the block is resting at the equilibrium position, you quickly kick the block (bead) so that it moves to the right with a speed v.

(**d**) Immediately after the block is kicked, what is the total kinetic energy of the system?

(**e**) For this kicked block, how far to the right (value of x) does the block reach before stopping to turn around?

Now, let's go back to the set up before the block (bead) is kicked to the right. The block (bead) is resting at the equilibrium position that you calculated in (b). Now, you suddenly turn off the laser light. In our model, this means that we cut the spring on the right (i.e. we suddenly set $k_{trap} = 0$).

(**f**) Immediately after the right spring is cut, what is the acceleration (magnitude and direction) and the velocity (magnitude and direction) of the block?

(**g**) Immediately after the right spring is cut, what is the total energy of the system?
[*Answer*: $k_{DNA}L^2\frac{(1-k)^2}{2}$]

(**h**) You should find in (g) that the total energy of the system after turning off the laser light is less than the total energy before turning off the laser light (or cutting the right spring). So is the law of energy conservation violated here? If not, where did the energy "disappear" to?

Problem 4. Gravity and Taylor approximation

According to Newton, a body of mass m_1 attracts a body of mass m_2 to itself with a force

$$\vec{F} = -\frac{Gm_1 m_2}{r^2}\hat{r} \tag{1}$$

where G is a positive constant called the Newton's gravitational constant, and $\hat{r}$ is the unit vector that starts from the "source" object and terminates at the "target" object ("unit vector" = vector of length 1; we use the little hat ˆ to denote unit vectors) (Fig. 4A). When the objects are no longer point-sized but instead have finite (i.e. non-zero) volume, then computing the gravitational force of attraction between the two objects becomes more complicated. Specifically, you need to break up the objects into little tiny pieces, compute the attraction from each tiny piece, then vectorially sum the forces due to each tiny piece. It turns out that if you have objects of "nice" shape and mass distribution, then a simple rule governs the gravitational force between those objects (we will see this when we discuss "**center of mass**" later in this course). One example is a sphere with a uniform mass density (i.e. the total mass of the sphere is uniformly distributed all over, so the density is the same everywhere in the sphere). Let's treat our Earth as a sphere of radius R and mass M that is uniformly distributed over itself (Fig. 4B). Suppose there's also another spherical object, call it a particle, of mass m whose size is negligible (so it's basically a point object). This particle is at a height h above the *surface* of the Earth. The particle's position relative to the center of the Earth is described by the position vector $\vec{r}$ that starts from the center of the Earth and terminates at the particle's location. Let $r = |\vec{r}|$ be the length of this position vector. Note that $r = R + h$. For these two "nicely" shaped objects, the Earth exerts a gravitational force on the particle equal to

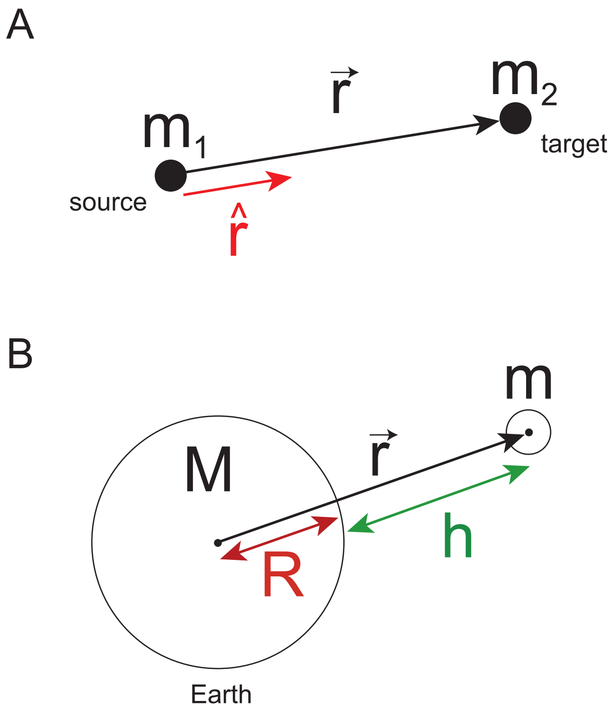

Figure 4: (A) Newton's law of gravity, (B) Gravitational attraction between the Earth and an object

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \tag{2}$$

(**a**) Show that you can write the magnitude of the gravitational force as

$$F = \frac{GMm}{R^2(1+\frac{h}{R})^2} \tag{3}$$

Note that in our everyday life, h is much smaller than R. (i.e. $h << R$). In this case, we can use an equation that looks much simpler than the one above to approximate the gravitational force. Let's derive this simpler equation step-by-step below.

(**b**) First, let's define a function

$$f(u) = \frac{1}{1+u} \tag{4}$$

Compute the Taylor series expansion of f(u) about zero. Recall that this is

$$f(u) = f(0) + uf^{(1)}(0) + u^2\frac{f^{(2)}(0)}{2!} + u^3\frac{f^{(3)}(0)}{3!} + ... \tag{5}$$

where $f^{(j)}(0)$ is the j-th derivative of f evaluated at $u = 0$. You can just compute the first four terms in the series as shown above. Then you can write "..." for the rest of the series in your answer.

(**c**) Now suppose we let $u = h/R$. We want to know what happens when u is a very small positive number (i.e. $|u| << 1$). Calculate the ratio of the 2nd order term in the Taylor series (the term that involves u^2) to the 1st order term in the Taylor series (the term that involves u). Then show that you can pick a value of $|u|$ to be so small that this ratio can be made very small. In other words, show that you can make the 2nd order term to be tiny compared to the 1st order term by making $|u|$ to be very small.

(**d**) Now calculate the ratio of the 3rd order term (the term that involves u^3) to the 2nd order term (the term that involves u^2). Show that this ratio can be made close to zero by picking $|u|$ to be a very small number.
[*Answer*: Ratio $= \frac{-4u}{3}$]

(**e**) Combining the results of (c) and (d), explain why the 3rd order term in the Taylor series (the term that involves u^3) is much much smaller than the 1st order term in the Taylor series (the term that involves u).

(**f**) In fact, you can extend above reasoning to show that *all* terms that are higher than 1st order in the Taylor series are very small compared to the 1st order term. Thus if we don't care too much about being precise to atomic accuracy, we can ignore all those terms and only keep the "leading order" terms, which are the 0th order and the 1st order terms. In other words, for $|u| << 1$, we can write

$$f(u) \approx f(0) + uf^{(1)}(0) \tag{6}$$

For $u = h/R$, we certainly have $|u| << 1$. Show that we can approximate the gravitational attraction force that you found in (a) as

$$F \approx \frac{GMm}{R^2}(1 - \frac{2h}{R}) \tag{7}$$

This result shows that as you go above the Earth's surface, but still remaining close the Earth's surface, the gravitational force decreases linearly as a function of the altitude h.

(**g**) Now, suppose you're right at the surface of the Earth ($h = 0$). Calculate the numerical value for the acceleration of the particle due to the force of gravity (here, you didn't have to use Taylor approximation. You could just use the original formula that you found in (a), and set $h = 0$ there. But the Taylor approximated formula in (f) also works here too). You can use $G = 6.67 \times 10^{-11} Nm^2/kg^2$, $M = 5.97 \times 10^{24} kg$, and $R = 6.37 \times 10^6 m$. [*Answer*: $g \approx 9.8 m/s^2$]

So you see, the reason that we have been using mg as the force of gravity is because we just assum $h = 0$ (i.e. h/R is so small that it's basically zero). But this is a very crude approximation.