NB1140: Physics 1A - Classical mechanics and Thermodynamics
Problem set 3 - Gravity, center of mass, and conservation of linear momentum

Week 3: 28 November - 2 December 2016

Problem 1. Tidal force

In this problem, we calculate the **tidal force** on an object. On Pg. 157 of the book, you can read about the concept behind "tidal force". In this problem, we calculate this force. Main idea: Each part of the Earth is gravitationally pulled to the moon by different amounts because each part of the Earth is a different distance from the moon. The same is true for the gravitational attraction of the Earth to the moon. These differing pulls cause the ocean to bulge around the spherical Earth. As we're standing on the Earth, we (and the Earth) rotate while the Ocean is not. This causes the bulge to look to us as a tide.

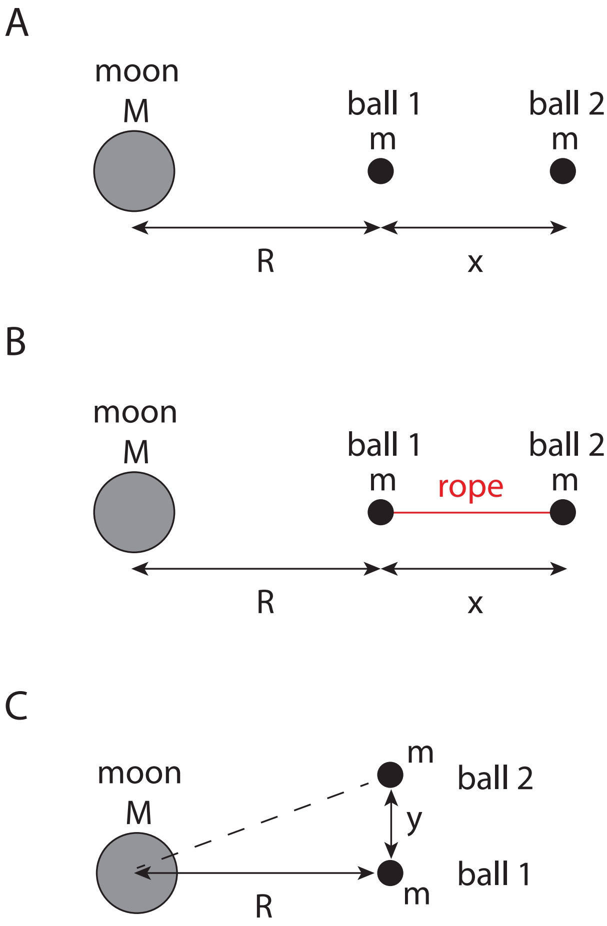

Figure 1: (**A**) Two balls of mass m each are gravitationally attracted towards the moon of mass M in space. (**B**) Difference in gravitational pull on ball 1 and ball 2 creates a tension in a massless rope that joins the two balls. Here, the two balls move together as if they were a single object. (**C**) Ball 1 and the center of the moon lie on the x-axis while ball 2 lies along the y-axis with ball 1.

(**a**) Consider three objects: a moon with a mass M and two identical balls of mass m each (Figure 1A). The two balls are separated by a distance x. Ball 1 is at a distance R from the center of the moon. Calculate the total gravitational force that ball 1 experiences and the total gravitational force that ball 2 experiences. Indicate the direction of the force one each ball (here, you can assume that the gravitational pull by the moon is much larger than the gravitational pull that the balls exert on each other).

(**b**) Let's ignore the gravitational force that the balls exert on each other. What is the difference in the forces that the moon exerts on ball 1 an ball 2? Get the exact expression.

(**c**) For the difference in the gravitational force in (b), let us assume that $x << R$. With this assumption, use Taylor approximation of $1/(1 + x/R)^2$, ignoring all terms that are higher than x/R (i.e. quadratic, cubic, quartic, etc.). From this approximation, show that

$$\frac{GMm}{R^2} - \frac{GMm}{(R+x)^2} \approx \frac{2GMmx}{R^3} \tag{1}$$

Hint: Look at the Taylor approximation in the last problem of Problem set 2.

This *difference in force* is called a **tidal force**. Note that it's not a *real* force but really a *fictitious* force. The real forces are the forces of gravity. But this difference in force is what you would feel if you were a ball 1 or a ball 2 falling towards the moon (or towards an Earth). When you are falling down (or diving), you feel "weightless". But this doesn't mean that there is no net force acting on you. Gravity is pulling you downwards to the ground. That's why you are falling down in the first place. But what you actually *feel* is something that goes *against* the force of gravity. For example, if you are standing on the ground, you feel being pushed up by the ground that's preventing you from falling through the ground (i.e. the ground pushes you up while the gravity is pushing you down). When you are in a free fall, there is no force counteracting the gravity. That is why you don't "feel" any force acting you. Note that the tidal force decreases as $1/R^3$ and increases linearly as a function of the distance of separation between the two balls.

(**d**) To understand how the balls would feel such a "counteracting force", suppose a massless rope connects the two balls while they are falling together towards the moon (Fig. 1B). Everything else is the same as the previous set up (Fig. 1A). What is the tension in this rope to first order in x (ignoring all higher order terms of x)? Here you can assume that the two balls will move together as one object (they are both falling towards the moon).
[*Hint*: By "falling together", we mean that the two balls move together (i.e. accelerate together) as if they were a single body, like in Problem 1 of Problem set 2. First, calculate the acceleration of the two conjoined bodies (total mass $2m$) and then compute the tension in the rope. Note here that the total force on the conjoined body is $\frac{2GMm}{R^2} - \frac{2GMmx}{R^3}$].
[*Answer*: $T = \frac{GMmx}{R^3}$]

This tension is what ball 1 and ball 2 would each feel. In other words, if you were "ball 1" and your friend were "ball 2", and the two of you were joined by a massless rope and falling towards the moon (or the Earth, if we were to replace the moon with the Earth), then you would feel the tension between the two of you but not the force of gravity (remember, when you fall freely, you feel no force).

(**e**) Now suppose we have a triangular configuration in which the moon and ball 1 are at the opposite ends of a right-angled triangle's base (horizontal line) while ball 1 and ball 2 are at the opposite ends of the vertical line of the right-angled triangle (Figure 1C). The two balls are separated by a distance y (Fig. 1C). The moon's center is at a distance R from ball 1. Assume that $y << R$. The force that the moon exerts on ball 2 is $\frac{GMm}{R^2+y^2}$. Ignore the gravitational pull of ball 1 by ball 2 and vice versa. Then with Taylor approximation (excluding all terms of 2nd order in y/R and higher), show that the force that the moon exerts on ball 2 is

$$\frac{GMm}{R^2+y^2} \approx \frac{GMm}{R^2} \tag{2}$$

(**f**) In (e), we see that to first order in y/R, the moon exerts the same gravitational force on ball 1 as it does on ball 2. The only difference is in the direction of the moon's gravitational pull. Let $\vec{F_1}$ be the gravitational force that the moon exerts on ball 1 and $\vec{F_2}$ be the gravitational force that the moon exerts on ball 2. In (e), we showed that $|\vec{F_1}| \approx |\vec{F_2}|$. Now, compute the difference in the x-component (horizontal component) of the two forces and the difference in the y-component (vertical component) of the two forces by using Taylor series that excludes all terms of order $(y/R)^2$ or greater.
Answer:

$$\vec{F}_{1,x} \approx \vec{F}_{2,x} \tag{3a}$$

$$\vec{F}_{1,y} - \vec{F}_{2,y} \approx -\frac{GMmy}{R^3}\hat{y} \tag{3b}$$

where $\hat{y}$ is a unit vector that points vertically upwards (i.e. vector of length one that starts from ball 1 and ends at ball 2). [*Hint*: If θ is the angle between the line that joins the moon and ball 1 and the line that joins the moon and ball 2, then $sin(\theta) \approx y/R$ to first order in y.]

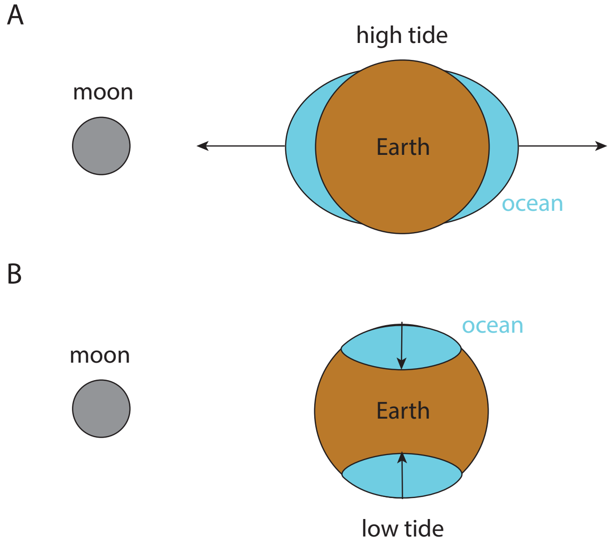

Figure 2: (**A**) Two high tides in a day. The two arrows indicate the stretching of the ocean surface *relative* to the person standing on a shoreline on Earth. (**B**) Two low tides in a day. The two arrows indicate the compression of the ocean surface *relative* to the person standing on a shoreline on Earth.

(**g**) Equation 3b is the tidal force, which points along the y-axis, for a scenario in which the two balls are arranged along the y-axis with the moon on the x-axis. Combing the tidal force along the x-axis (equation 1) and the tidal force along the y-axis (equation 3b), we can now understand how the ocean's tides rise high and fall low. The key is to think of one ball as a point on the surface of the Earth and the other ball as a point at the bottom of the ocean (on the ocean floor). Since the ocean water can flow and slosh around, independent of the solid Earth, the ocean surface be elongated outward towards the moon (Fig. 2). As you stand on the shoreline (part of solid Earth like the ocean floor), you rotate with the Earth while the ocean water is free to move about. Relative to you, it's the ocean water that's moving relative to you. Based on your answers from the previous parts, explain in words why there would be two high tides and two low tides per day (i.e. explain Figure 2).

Problem 2. Synchronous orbit of two satellites around the Earth

Consider two identical of mass m orbiting the Earth (Figure 3). The Earth has mass M. We treat the Earth as a sphere of uniform mass density. Each satellite is at a distance R from the center of the Earth. The two satellites are at diametrically opposite ends of a circle that they orbit in (Figure 3). Each satellite orbits around the Earth and the other satellite at speed v.

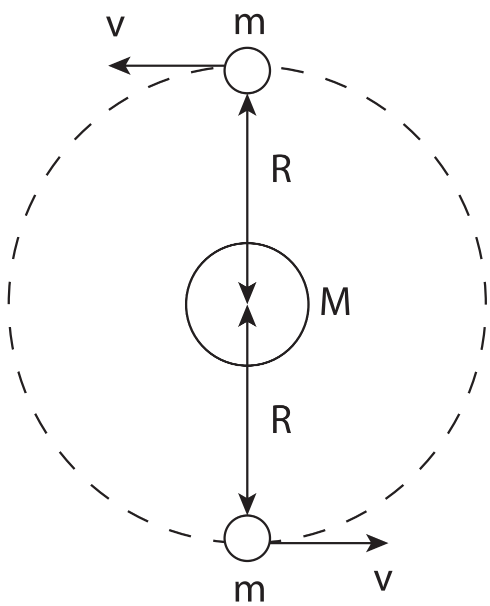

Figure 3: *A planet of mass m orbits around a planet of mass M and another planet of mass m. The other planet of mass m is synchronously orbiting around the other two planets. Both m's orbit at speed v in a circle of radius R. The planet of mass M is at the center of this circle and the distance between the center of M and centers of each m is R.*

(**a**) What is the centripetal force on each satellite (both the direction and magnitude)?

(**b**) What is net the force acting on the Earth?

(**c**) Show that the **period** T of each satellite's circular orbit (i.e. the time taken for each satellite to go around the circle once) is

$$T = 4\pi\sqrt{\frac{R^3}{G(4M+m)}} \tag{4}$$

(**d**) For the situation described here (Figure 3), the Earth remains stationary. If the Earth moves, then the motion of the whole system (the satellites and the Earth) would get quite complicated. Suppose you nudge (push) the Earth towards one of the two satellites at some instant while the two satellites are in orbit. Describe in words what the subsequent motion of the two satellites and the Earth would be. Based on this, is the synchronous orbit of the two satellites stable or unstable?

(**e**) Suppose that we replace one of the satellites with a different satellite that has a mass $m + \delta m$. We keep the other satellite as it is (it still has mass m). Can such a system sustain a synchronous orbit of the two satellites and the Earth remaining still at the center?

This problem gives you an idea of how difficult it is to keep tens to hundreds of satellites in orbit around the Earth (and any planet). To sustain satellites in orbit at different radii around a common planet, satellites have to constantly propel themselves (requires energy) to correct and adjust their orbit around the Earth. Otherwise, satellites can collides or fall into each other (and eventually fall to the ground!).

Problem 3. Gravitational energy of a complex configuration and work that you do to dismantle it

(**a**) Consider an empty space. You then bring a particle of mass m_1 from very far away (in fact, out from "infinity") to a certain position in space. How much work do you do in this process?

(**b**) Holding the first particle (of mass m_1) firmly in its place so that it remains at rest, gravitational pull of the first particle brings in a second particle, of mass m_2, from very far away (out from "infinity") to a position that is of distance r_{12} from the first particle. How much work does the gravity do in this process?

(**c**) Now while holding the two particles (of masses m_1 and m_2) firmly in their place so that they remain at rest, the gravitational pull of both particles bring a third particle, of mass m_3, from far away (out from "infinity") to a position that is of distance r_{13} from the first particle (of mass m_1) and of distance r_{23} from the second particle (of mass m_2). What is the total amount of work that gravity does to assemble this triangle of particles starting from empty space (from (a) to this part)?

(**d**) Suppose you dismantle the triangle in (c) by kicking each of the three particles out to infinity. How much work do you do on the system (system = three particles) in this process? [*Hint*: $W = \Delta U$, where W is the work that *you* do in a process and ΔU is the change in the potential energy of the system in the process].

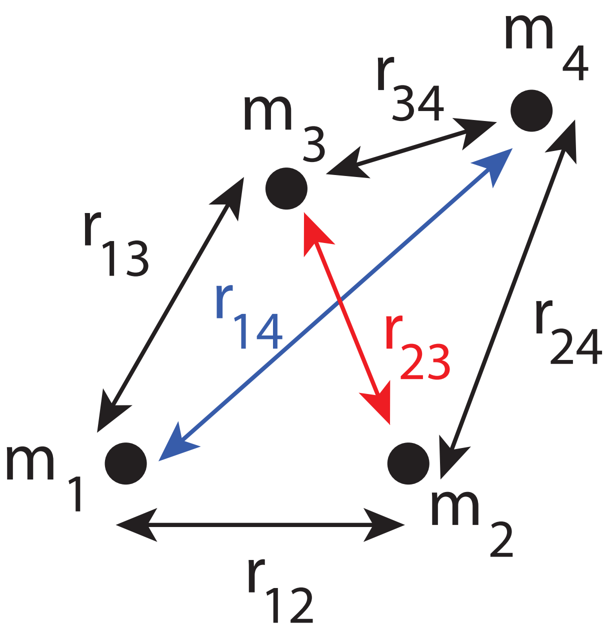

Figure 4: A polygon of four particles.

(**e**) Suppose you assemble a polygon consisting of four corners, with a particle at each corner of the polygon (Figure 4). The particles have masses m_1, m_2, m_3, and m_4. The distance between particle-i (of mass m_i) and particle-j (of mass m_j) is r_{ij}. Mimicking the process described in (a)-(c), show that the gravitational potential energy of this polygon of particles is

$$U = -\frac{1}{2}\sum_{i=1}^{4}\sum_{\substack{j=1\\ j\neq i}}^{4}\frac{Gm_im_j}{r_{ij}} \tag{5}$$

(**f**) What is the work that you do to push all the charges out to infinity starting from this polygonal configuration (Fig. 4) (i.e. the work that you do to dismantle the polygon by kicking each particle out to infinity)?

Problem 4. Calculating the center of mass

(**a**) Consider a sphere of radius R consisting of two solid hemispheres (Figure 5A). The northern hemisphere has total mass m_n that is uniformly distributed throughout the hemisphere. The southern hemisphere has total mass m_s that is uniformly distributed throughout the hemisphere. Let (x, y) = (0, 0) be the center of the sphere. Where is the center of mass? Calculate only the x-component of the position (calculating the y-component of the center of mass' position is more difficult and involves some integrals).
[*Answer*: $x = 0$]

(**b**) Consider a uniform sphere of radius R and mass M (Figure 5B). The mass M is uniformly distributed throughout the sphere. The center of this sphere is at (x, y)=(0,0). You carve a spherical hole of radius r_1 ($r_1 < R$) out of this full sphere. The center of the spherical hole coincides with the center of the original sphere (they are both at (x, y) = (0, 0)). Where is the center of mass of this new hollow sphere?
[*Answer*: $(x, y) = (0, 0)$]

(**c**) Consider a uniform sphere of radius R and mass M (Figure 5C). The mass M is uniformly distributed throughout the sphere. You carve a spherical hole of radius r ($r < R/2$) out of this full sphere. The center of the spherical hole is at (x, y) = (R/2, 0) and the center of the original sphere (before hole is made) is at (x, y)=(0,0). Where is the center of mass of this new hollow sphere?
[*Answer*: $(x, y) = (-\frac{R}{2}\frac{r^3}{R^3-r^3}, 0)$]

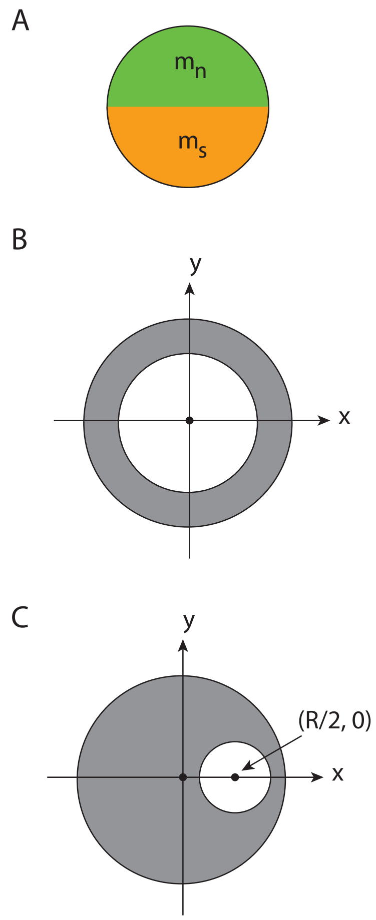

Figure 5: Spheres

Problem 5. Conservation of linear momentum and kinetic energy - (elastic collision)

A mass m with speed v approaches a stationary mass M. The masses bounce off each other elastically. What are the final velocities of the particles? Assume that the entire motion takes place in one dimension.
Answer:

$$v_m = \frac{(m - M)v}{m + M}, \qquad v_M = \frac{2mv}{m + M} \tag{6}$$

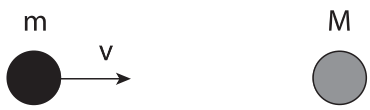

Figure 6: 1-dimensional collision

Problem 6. Conservation of linear momentum and motion of center of mass - (walking on a boat)

Consider a boat of mass M and length L floating and remaining stationary on a frictionless water. A person of mass m stands at the front of the boat. She then walks down the length of the boat to the back of the boat. When she reaches the back, she stops walking. Relative to the initial position, where is the final position of the boat on the water?
[*Answer* Distance of $\frac{mL}{M+m}$ to the right of the boat's initial position.]

Problem 7. Conservation of linear momentum but not kinetic energy - (Inelastic collision)

A continuous stream of sand drops vertically at a rate R (mass / time) onto a moving conveyor belt. Assume that just before landing on the belt, the speed of the sand is nearly zero (i.e. sand is dropped from just above the belt rather than being dropped from an appreciable height).

(**a**) What force must you apply to the conveyor belt to keep the belt moving at a constant speed v at all times? [*Hint*: By definition, the rate of change of momentum = force]
[*Answer*: $F = Rv$]

(**b**) How much kinetic energy does the sand gain per unit time? [*Hint*: Just before hitting the belt, the kinetic energy of each sand grain is zero]
[*Answer*: $Rv^2/2$]

(**c**) How much work do you do per unit time? [Note: this is also called the **power**]
[*Answer*: Rv^2]

(**d**) How much energy is lost to heat per unit time? [*Hint*: The total energy is conserved in any process. Here, any energy that is *missing* in the kinetic energy compared to the work that you put in must go away into heat]
[*Answer*: $Rv^2/2$]