NB1140: Physics 1A - Classical mechanics and Thermodynamics
Solution to Quiz 2
Wednesday 7 December 2016

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(**a**) At the equilibrium position, the force acting on the bead due to the DNA is counteracted (cancelled) exactly by the pull of the optical trap spring. Both springs would be stretched. The DNA spring is stretched by $L + x$ (true for both when $x < 0$ and when $x > 0$) and the trap spring is stretched by $|x|$. Now we know that in this case, if we let x be the equilibrium position, then $x < 0$ because when $x = 0$, the DNA is stretched while the trap spring doesn't exert any force on the bead. Picking the right side (positive direction of x-axis) to be positive and the left side (negative direction of x-axis) to be negative, we have

$$\begin{aligned} -k_{DNA}(L + x) - k_{trap}x &= 0 \\ \implies x &= \frac{-k_{DNA}L}{k_{DNA} + k_{trap}} \end{aligned} \tag{1a}$$

This is the equilibrium position. The DNA spring was already stretched by a distance L when the bead was at the origin of the x-axis. So when the bead is at this equilibrium position, the DNA spring is stretched by $L + x$ (note that $x < 0$ here indeed from the minus sign in $\frac{-k_{DNA}L}{k_{DNA}+k_{trap}}$). So we have

$$\begin{aligned} L + x &= L + \frac{-k_{DNA}L}{k_{DNA} + k_{trap}} \\ &= L(1 - \frac{k_{DNA}}{k_{DNA} + k_{trap}}) \end{aligned} \tag{2a}$$

This is the total stretched distance of the DNA spring when the bead is at its equilibrium position.

(**b**) The total energy E_{total} is just the sum of the two spring potential energies (no kinetic energy because the bead is not moving). So we have

$$E_{total} = \frac{k_{DNA}(L + x)^2}{2} + \frac{k_{trap}x^2}{2} \tag{3}$$

Defining a new variable k to be $k = \frac{-k_{DNA}}{k_{DNA}+k_{trap}}$, we have

$$x = -Lk \tag{4}$$

and thus we have

$$
\begin{aligned}
E_{total} &= \frac{k_{DNA}(L - kL)^2}{2} + \frac{k_{trap}k^2L^2}{2} \\
&= \frac{k_{DNA}L^2(1 - k)^2}{2} + \frac{k_{trap}k^2L^2}{2} \\
&= \frac{L^2}{2}(k_{DNA}(1 - k)^2 + k_{trap}k^2) \qquad (5a)
\end{aligned}
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This is the total energy when the bead is at its equilibrium position and is not moving.