NB1140: Physics 1A - Classical mechanics and Thermodynamics
Solution set 1 - Describing motion of objects (Kinematics)
Week 1: 14- 18 November 2016

Solution to Problem 1. Squished between two trains (Zeno's paradox)

(a) There are at least two ways to calculate d(t).

Method 1: First, we know that at $t = 0$, we have $d(0) = D$. This distance decreases over time because the two trains are getting closer to each other. So at later times, the initial distance D is "taken away" by the two trains. One train takes away a distance Vt and the other also takes away a distance Vt. So a total of $2Vt$ is taken away from D. So we have

$$\begin{aligned} d(t) &= D - Vt - Vt \\ &= D - 2Vt \end{aligned} \tag{1a}$$

Method 2: Think of relative motions (in chapter 3). Suppose you're standing inside the left train. To you, the left train is not moving. The right is moving towards you at speed $2V$ (we say that the "right train is moving at a speed $2V$ with respect to the left train", or "right train is moving relative to the left train at a speed $2V$). This means that after time t, the train is closer to you by a distance $2Vt$ and $d(0) = D$. So we have

$$d(t) = D - 2Vt \tag{2}$$

(b) When the two trains meet at time t_f, the distance between the two trains is zero. So we have $d(t_f) = 0$. Solving for t_f in this equation, we get

$$\begin{aligned} d(t_f) = 0 &\implies 0 = D - 2Vt_f \\ t_f &= \frac{D}{2V} \end{aligned} \tag{3a}$$

(c) Each train travels at a constant speed V for time t_f. Thus the total distance that each train moves through is Vt_f:

$$Vt_f = V\frac{D}{2V} \implies \text{total distance travelled by each train} = \frac{D}{2} \tag{4}$$

Note that above answers don't depend on u because how the trains move is independent of what the fly is doing.

(d) The answer must be $\frac{D}{2}$ because the two trains are the exact mirror images of each other. In other words, if you flip the diagram left to right (i.e., flip the page along its long edge), then you wouldn't see that anything has changed (the picture would look exactly the same to you). So if the trains were to meet at a point that's not exactly halfway between the two, then one train must have moved more than the other during time t_f. But this cannot be because that means flipping the page should give you a different picture. And we know that this doesn't happen. So, indeed the answer must be $\frac{D}{2}$; the two trains meet exactly halfway between their initial positions.

(e) The fly is always moving at a constant speed u (but note that it does *not* move at a constant velocity at all times because the fly turns around when it reaches one of the trains). The fly is always moving at the speed u because we assume that the fly takes zero time to turn around. The fly is moving as long as the trains give it room to move around (i.e. until the two trains collide). So the fly is moving between $t = 0$ and $t = t_f$. So the total distance that it moves is

$$ut_f = u\frac{D}{2V} \implies \text{total distance flown by the fly} = \frac{uD}{2V} \tag{5}$$

This distance does *not* depend on the initial position of the fly. You can see this from the fact that we didn't need to worry about where the fly is initially in the argument above. The reason that the answer doesn't depend on the fly's initial position is that the distance is just the speed times the total time taken for the motion, and the fly is always able to move at this constant speed u. And the subtle reason that this can happen is that $u > V$. If $u < V$, then the answer would depend on the fly's initial position because one of the trains would hit and push the fly from the fly's back *before* the two trains meet. In this case, the fly would no longer be flying at the constant speed u but would be pushed at a constant speed V by one of the trains from its back.

(f) In its first one-way flight during time interval Δt_1), the combined distance flown by the fly and the distance moved by the right train must equal D because this is the distance between the fly and the right train at $t = 0$ and the two must meet after time Δt_1. So we have

$$\begin{aligned} D &= u\Delta t_1 + V\Delta t_1 \\ \implies \Delta t_1 &= \frac{D}{u+V} \end{aligned} \tag{6a}$$

The distance d_1 flown by the fly during this time Δt_1 is

$$d_1 = u\Delta t_1 \implies d_1 = \frac{uD}{u+V} \tag{7}$$

(g) During its second one-way flight (which takes time Δt_2), the combined distance flown by the fly and the distance moved by the left train must equal $(u - V)\Delta t_1$ because this is the distance between the fly and the left train after the first one-way trip. The fly and the left train must meet after time Δt_2 by closing this distance D_2. These two facts lead us to the following two equations

$$D_2 = (u - V)\Delta t_1 \tag{8a}$$
$$D_2 = (u + V)\Delta t_2 \tag{8b}$$

and thus

$$\Delta t_2 = \frac{u-V}{u+V}\Delta t_1 \tag{9}$$

and

$$\begin{aligned} d_2 &= u\Delta \\ &= u\frac{u-V}{u+V}\Delta t_1 \\ &= u\frac{u-V}{u+V}\frac{D}{u+V} \end{aligned} \tag{10a}$$

(h) Let's follow the suggested steps to derive d_n. First, we note that the distance D_n between the incoming train and the fly at the end of the (n-1)st one-way trip is

$$D_n = (u-V)\Delta t_{n-1} \tag{11}$$

due to the same argument as the one we used in part (g). Also by the same argument as in part (g), we have

$$D_n = (u+V)\Delta t_n \tag{12}$$

Above is because the fly and the incoming train must meet (i.e. fully close the gap, which has distance D_n) by the end of the n-th one-way flight. From above two equations, we can solve for Δt_n in terms of Δt_{n-1}:

$$\Delta t_n = \frac{u-V}{u+V}\Delta t_{n-1} \tag{13}$$

and thus

$$\begin{aligned} d_n &= u\Delta t_n \\ &= u\frac{u-V}{u+V}\Delta t_{n-1} \end{aligned} \tag{14a}$$

And by recursion (induction) relation, we get

$$\begin{aligned} \Delta t_n &= \frac{u-V}{u+V}\Delta t_{n-1} \\ &= \frac{u-V}{u+V}\frac{u-V}{u+V}\Delta t_{n-2} \\ &= \left\{\frac{u-V}{u+V}\right\}^3\Delta t_{n-3} \\ &\dots \\ &= \left\{\frac{u-V}{u+V}\right\}^{n-1}\Delta t_1 \\ &= \left\{\frac{u-V}{u+V}\right\}^{n-1}\frac{D}{u+V} \end{aligned} \tag{15a}$$

And so equation (14a) becomes

$$d_n = u\Big\{\frac{u-V}{u+V}\Big\}^{n-1}\frac{D}{u+V} \tag{16}$$

(i) We want to compute the following *infinite geometric sum*

$$\begin{aligned} d_{total} &= \sum_{n=1}^{\infty} d_n \\ &= \sum_{n=1}^{\infty} u\Big\{\frac{u-V}{u+V}\Big\}^{n-1}\frac{D}{u+V} \\ &= \frac{uD}{u+V}\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} \end{aligned} \tag{17a}$$

In the last line above, we took the $\frac{uD}{u+V}$ (which is just the d_1) outside the summation because it's a constant that doesn't depend on n. The stuff inside the summation sign is an infinite geometric sum. You learned to compute this in your analysis course. First note that we have

$$\frac{u-V}{u+V} < 1 \tag{18}$$

So the sum *converges* (i.e. the infinite sum gives a number instead of blowing up to an infinity). This is also a fact that you learned in your analysis course. You have two options to compute the sum: (1) just use the formula that you learned in your analysis course, or (2) (the better option) derive this formula yourself. Let's take option (2). Let's rewrite the sum in equation (17a):

$$\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} = 1 + \frac{u-V}{u+V} + \Big\{\frac{u-V}{u+V}\Big\}^2 + \Big\{\frac{u-V}{u+V}\Big\}^3 + ... \tag{19}$$

If we define $r = \frac{u-V}{u+V}$, then above equation is just

$$\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} = 1 + r + r^2 + r^3 + ... \tag{20}$$

Multiplying both sides of above equation by r, we get

$$r\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} = r + r^2 + r^3 + r^4.... \tag{21}$$

But the right hand side of above equation is actually just 1 less than the infinite sum that we're after in the first place:

$$r\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} = \sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1} - 1 \tag{22}$$

Now, we can move the terms around in the above equation to isolate and then solve for $\sum_{n=1}^{\infty}\Big\{\frac{u-V}{u+V}\Big\}^{n-1}$. This gives us

$$\sum_{n=1}^{\infty}\left\{\frac{u-V}{u+V}\right\}^{n-1}=\frac{1}{1-r} \tag{23}$$

The right hand side of above equation is exactly what you learned in your analysis course. To finish this problem, we go back to equation (17a):

$$\begin{aligned}
d_{total} &= \frac{uD}{u+V}\sum_{n=1}^{\infty}\left\{\frac{u-V}{u+V}\right\}^{n-1} \\
&= \frac{uD}{u+V}\frac{1}{1-r} \\
&= \frac{uD}{u+V}\frac{1}{1-(u-V)/(u+V)} \\
&= \frac{uD}{u+V}\frac{u+V}{2V} \\
&= \frac{uD}{2V}
\end{aligned} \tag{24a}$$

This is exactly what you found in part (e) (equation (5)). We have used two different methods, one that's easier and one that's harder, to get the same answer (as we should!).

We computed a sum of infinite number of d_n's instead of a finite number of them because the fly has to take an infinite number of turns to travel this finite distance d_total. To see that, note that in part (h), we cannot have $D_n = 0$ (equation 11) because $u > V$ and thus Δt_n cannot be zero (because there's always a non-zero gap distance between the fly and the incoming train).

(j) This situation wouldn't work for a real fly even if the real fly could instantaneously turn around after reaching a train for its next one-way flight. This is because a real fly is *not* a point object. It has a non-zero volume. So the D_n cannot keep on getting smaller towards zero and have $n \to \infty$. The fly is already dead at some finite value of n. To be concrete, if the fly is a sphere with diameter h, then from equation (16), we see that $d_n < h$ when we have

$$u\left\{\frac{u-V}{u+V}\right\}^{n-1}\frac{D}{u+V} < h \tag{25}$$

Solution to Problem 2. Why bugs spiral into light (logarithmic spiral)

(a) Let $|\vec{V}_{||}| = V_{||}$ and $|\vec{V}_{\perp}| = V_{\perp}$ (the lengths of these two vectors). From the right-angled triangle in figure 2, we get

$$V_{||} = Vcos(\theta) = Vcos(\frac{\pi}{2}) = 0 \tag{26a}$$

$$V_{\perp} = Vsin(\theta) = Vsin(\frac{\pi}{2}) = V \tag{26b}$$

Another way to get this is visually: When $\theta = \frac{\pi}{2}$, we don't actually have a triangle. The $\vec{V_{||}}$ disappears and we have $\vec{V} = \vec{V_{\perp}}$.

(b) The bug moves around the circle of radius R at a constant speed V (but *not* at a constant velocity). In fact, the bug performs a uniform circular motion with a centripetal acceleration vector $\vec{a}$ whose length remains constant at $\frac{v^2}{R}$ over time. The bug will never fall into the lamp.

(c) Let $|\vec{V_{||}}| = V_{||}$ and $|\vec{V_{\perp}}| = V_{\perp}$ (the lengths of these two vectors). From the right-angled triangle in figure 2, we get

$$V_{||} = Vcos(\theta) = Vcos(0) = V \tag{27a}$$

$$V_{\perp} = Vsin(\theta) = Vsin(0) = 0 \tag{27b}$$

Another way to get this is visually: When $\theta = 0$, we don't actually have a triangle. The $\vec{V_{\perp}}$ disappears and we have $\vec{V} = \vec{V_{||}}$.

(d) The bug no longer moves in a circle. It moves along a straight line (radial line) joining the lamp and the bug itself. It moves along this radial line at a constant speed $\vec{V_{||}} = V$. Since $r(0) = R$ and the bug gets closer and closer to the lamp over time, we know that r(t) must decrease over time until it becomes zero (that's when the bug and the lamp meet). In fact, we have

$$r(t) = R - Vt \tag{28}$$

The bug hits the lamp when $r(t) = 0$. From above equation, we can solve for t:

$$0 = R - Vt$$

$$\implies t = \frac{R}{V} \tag{29a}$$

So the bug arrives at the lamp at time $t = R/V$.

(e) *Solution will be given after the quiz.*

(f) *Solution will be given after the quiz.*

(g) To get $\phi(t)$ in radians, we will use the fact that at given instant of time t, the *instantaneous angular velocity* $d\phi/dt$ is given by

$$r(t)\frac{d\phi}{dt} = V_{\perp} \tag{30}$$

We will derive this result later in our course when we talk about angular motion. Plugging into $V_{\perp}$ what we found in part (e), we get

$$\begin{aligned}
r(t)\frac{d\phi(t)}{dt} &= Vsin(\theta) \\
\implies \frac{d\phi(t)}{dt} &= V\frac{sin(\theta)}{r(t)} \\
\implies \frac{d\phi(t)}{dt} &= \frac{Vsin(\theta)}{R - Vtcos(\theta)} \\
\implies d\phi &= \frac{Vsin(\theta)}{R - Vtcos(\theta)}dt \\
\implies \int_0^{\phi} d\phi &= \int_0^t \frac{Vsin(\theta)}{R - Vtcos(\theta)}dt \\
\implies \phi &= Vsin(\theta)\frac{log(R - Vtcos(\theta))}{-Vcos(\theta)}\Bigg|_0^t \\
\implies \phi &= -tan(\theta)(log(R - Vtcos(\theta)) - log(R)) \\
\implies \phi &= -tan(\theta)log\left(\frac{R - Vtcos(\theta)}{R}\right) \qquad (31a)
\end{aligned}$$

Here, the log is the natural logarithm (base e; sometimes written as "ln"). Note that this is in the form $\phi(t) = A \cdot log(B(t))$, with

$$A = -tan(\theta) \tag{32a}$$

$$B(t) = \frac{R - Vtcos(\theta)}{R} \tag{32b}$$

Solution to Problem 3. Travelling faster than light

(a) Note that $d \cdot tan(\theta)$ is the height of the yellow triangle. At this moment, the fly moves up at speed u and thus increases the length of the yellow triangle's height by rate u (measured in length / time). If we wait a long time, then the base of the yellow triangle will also move because the fly will have moved so far up. But here, we are considering what happens at the next immediate (infinitesimal) moment in time later. In this case, we can ignore that the base of the triangle also moves up. So we have

$$\begin{aligned} u &= \frac{d(d \cdot tan(\theta))}{dt} \\ \implies u &= d\frac{d(tan\theta)}{dt} \\ \implies \frac{d(tan\theta)}{dt} &= \frac{u}{d} \end{aligned} \tag{33a}$$

(b) From the geometry of the big right-angled triangle in figure 3, we have

$$\begin{aligned} \text{height of the big triangle} &= (d+L)tan\theta \\ \implies \frac{d(\text{height of the big triangle})}{dt} &= (d+L)\frac{d}{dt}(tan\theta) \\ \implies v &= (d+L)\frac{u}{d} \end{aligned} \tag{34a}$$

(c) Let c be the speed of light. We want $v > c$. From above equation, this means

$$\begin{aligned} \frac{(d+L)u}{d} &> c \\ \implies 1 + \frac{L}{d} &> \frac{c}{u} \\ \implies \frac{L}{d} &> \frac{c}{u} - 1 \end{aligned} \tag{35a}$$

(d) Plugging in the numbers: $u = 1000km/h = 10^6 m/3600s$. Thus

$$\frac{c}{u} - 1 = \frac{3 \times 10^8 m/s}{10^6 m} \cdot (3600s) - 1 \approx 11 \times 10^5 - 1 \approx 10^5 \tag{36}$$

In the last part of above equation, we ignore the "1" because it's so tiny compared to 10^5. Now d=1mm. So using equation (37a), we have

$$L \approx 10^5 \cdot 10^{-3} m = 10^2 m = 100m \tag{37}$$

(Note that the inequality ">" becomes an equality "=" because we get the lower bound of the inequality for the speed of light.)

Thus when $L \approx 100m$, the shadow of the mini 747 will move at the speed of light. Note that if the wall is further way than 100 m, the shadow will travel faster than the speed of light.

Solution to Problem 4. Bacterial chemotaxis: Random walk

(a) The graphs of position VS time and velocity VS time are in Figure 1 on the right. Watching over a long time, the average displacement is $\frac{VT}{2}$ (we can see this from the position VS time graph). The half-way between the two extreme positions, 0 and VT, is $\frac{VT}{2}$. If you watch over a long time, you find that the E. coli spends an equal amount of time at positions $x < \frac{VT}{2}$ and at $x > \frac{VT}{2}$. Another acceptable answer is that the E. coli doesn't go anywhere over time. The $\frac{VT}{2}$ is just a constant offset relative to the origin. It's not important. The important point is that the E. coli is stuck in this region.

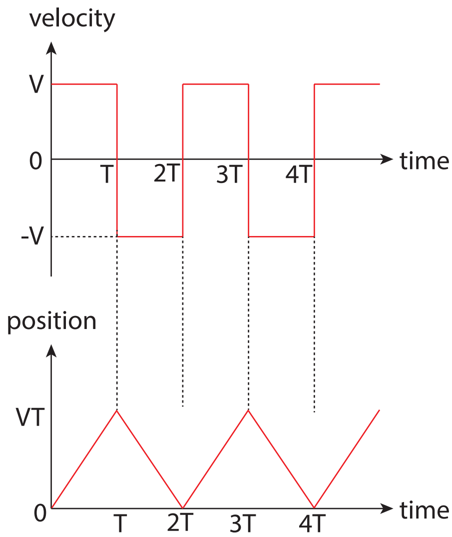

Figure 1: Solution to problem 4(b)

(b) The graphs of position VS time and velocity VS time are in Figure 2 on the right. Over time, the E. coli "drifts" to the right on the line that it moves on (i.e. in the positive direction on the x-axis). This drift in the +x direction is due to the longer travel time in the positive direction than in the negative direction (this model is very simple but this is indeed the central element of real (more complicated) chemotaxis of E. coli). We can approximate the average displacement relative to the initial position by computing the average drift velocity V_{avg}:

$$V_{avg} = \frac{VT_R - VT_L}{T_R + T_L} \tag{38}$$

Thus the average displacement relative to the E.coli's initial position is

$$\begin{aligned} d(t) &= V_{avg} \cdot t \\ &= \frac{VT_R - VT_L}{T_R + T_L} t \end{aligned} \tag{39a}$$

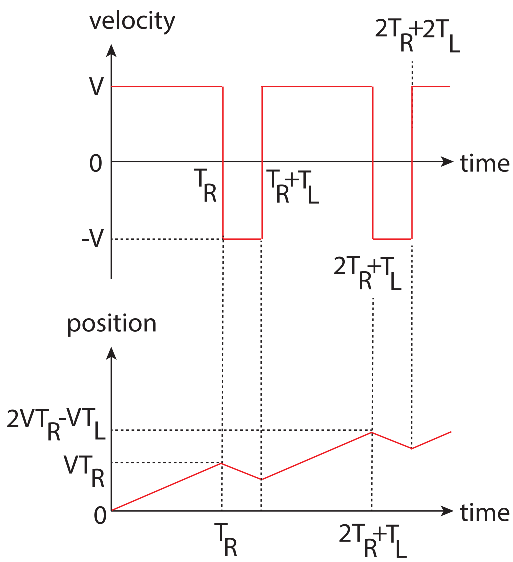

Figure 2: Solution to problem 4(a)