NB1140: Physics 1A - Classical mechanics and Thermodynamics
Solution set 3 - Gravity, center of mass, and conservation of linear momentum

Week 3: 28 November - 2 December 2016

Solution to problem 1.

(**a**) Ball 1 is pulled towards the moon and is also pulled towards ball 2. If we let the positive x-axis be towards ball 2 (and thus the negative x-axis point towards the moon, the total force on ball 1 is

$$F_{\text{ball 1}} = -\frac{GMm}{R^2} + \frac{Gm^2}{x^2} \tag{1}$$

The total force on ball 2 is due to the pull by ball 1 (in the negative x-direction) and the pull of the moon (in the negative x-direction):

$$F_{\text{ball 2}} = -\frac{GMm}{(R+x)^2} - \frac{Gm^2}{x^2} \tag{2}$$

If we assume that the moon exerts a larger gravitational attraction on each ball than the gravitational force that the balls exert on each other, then $F_{\text{ball 1}}$ is a negative number (so the net force on ball 1 is towards the negative x-axis) and $F_{\text{ball 2}}$ is also a negative number (so the net force on ball 2 is towards the negative x-axis).

(**b**)

$$F_{\text{ball 1}} - F_{\text{ball 2}} = -\frac{GMm}{R^2} + \frac{GMm}{(R+x)^2} \tag{3}$$

This is a negative number because the force that the moon exerts on ball 1 is larger in magnitude than the magnitude of the force that the moon exerts on ball 2 (because ball 1 is closer to the moon than ball 2).

(**c**)

$$\begin{aligned}\frac{GMm}{(R+x)^2} &= \frac{GMm}{R^2(1+x/R)^2} \\ &\approx \frac{GMm}{R^2}(1-\frac{2x}{R}) \end{aligned} \tag{4a}$$

where we have used the Taylor approximation of $1/(1+u)^2$ (with $u = x/R$ around zero) and ignored all the very small terms (terms with order that's higher than u such as

u^2, u^3, u^4). So we have

$$\begin{aligned}
F_{\text{ball 1}} - F_{\text{ball 2}} &= -\frac{GMm}{R^2} + \frac{GMm}{(R+x)^2} \\
&\approx -\frac{GMm}{R^2} + \frac{GMm}{R^2}(1 - \frac{2x}{R}) \\
&= -\frac{2GMmx}{R^3} && \text{(5a)}
\end{aligned}$$

According to our sign convention that we introduced in (a), the negative sign means that this *differential force* points in the negative x-axis.

(**d**) A massless rope connects the two balls. The two balls move together as one object of mass $2m$. The total force on the system of two balls is $F_{\text{ball 1}} + F_{\text{ball 2}}$. This force must be equal to total mass of the system ($2m$) times the acceleration of the system a:

$$\begin{aligned}
F_{\text{ball 1}} + F_{\text{ball 2}} &= 2ma \\
\implies -\frac{GMm}{R^2} - \frac{GMm}{(R+x)^2} &= 2ma \\
\implies -\frac{GMm}{R^2} - \frac{GMm}{R^2}(1 - \frac{2x}{R}) &= 2ma \\
\implies -\frac{2GMm}{R^2} + \frac{2GMmx}{R^3} &= 2ma \\
\implies a = -\frac{GM}{R^2} + \frac{GMx}{R^3} & && \text{(6a)}
\end{aligned}$$

From our lesson on center of mass, note here that the total force on the system (system = 2 balls) is just the total *external force* on the system (the moon is external to the system). The force that ball 1 exerts on 2 is cancelled out by the force that ball 2 exerts on ball 1 (by Newton's 3rd law – these two are *internal forces*). Now, writing the Newton's 2nd law only for ball 1 (and ignoring the force that ball 2 exerts on ball 1 because we said in (a) that it will be much smaller than everything else), we obtain

$$F_{\text{ball 1}} + T = ma \tag{7}$$

Here, T is a positive number (and we put a positive sign in front of it because it points in the positive x-axis). $F_{\text{ball 1}}$ is already a negative number so we don't put a negative sign in front of it (it's a vector, so a number with a proper sign). So

$$\begin{aligned}
F_{\text{ball 1}} + T &= ma \\
\implies -\frac{GMm}{R^2} + T &= ma \\
\implies T &= -\frac{GMm}{R^2} + \frac{GMmx}{R^3} + \frac{GMm}{R^2} \\
\implies T &= \frac{GMmx}{R^3} && \text{(8a)}
\end{aligned}$$

(**e**) By Pythagorean theorem, the distance between the moon and ball 2 is $\sqrt{R^2 + y^2}$. Thus by Newton's law of gravity, the moon exerts gravitational pull on ball 2 of

$$F_{\text{ball 2}} = \frac{GMm}{R^2 + y^2} \tag{9}$$

Now, we expand above with Taylor series expansion of $1/(R^2 + y^2)$. We then ignore all terms that are higher order than y/R (assuming $y << R$). Doing this, we get

$$\begin{aligned} F_{\text{ball 2}} &= \frac{GMm}{R^2 + y^2} \\ &= \frac{GMm}{R^2(1 + (y/R)^2)} \\ &\approx \frac{GMm}{R^2}(1 - 2\frac{y^2}{R^2}) \\ &\approx \frac{GMm}{R^2} \end{aligned} \tag{10a}$$

where we have ignored the y^2/R^2 term because it is of higher order than y/R.

(**f**) Let θ be the angle of the apex of the triangle (as defined in the problem). Now, this θ must be very small (To be more precise, we say that $|\theta| << 1$; i.e. θ is much smaller than 1 radians; why it's one radians is not important here though). The point is that since θ is such a small angle, we expect $cos(\theta) \approx cos(0) = 1$. So the horizontal component of $\vec{F}_{\text{ball 2}}$ has magnitude equal to

$$F_{\text{ball 2}} cos(\theta) \approx F_{\text{ball 2}} = \frac{GMm}{R^2} \tag{11}$$

which is exactly the same as the force on ball 1 (which is entirely along the x-axis). So, the x-component of the force of gravity of the moon on ball 1 is approximately equal to the x-component of the force of gravity of the moon on ball 2.

Now, let's look at the y-component. Note that

$$\begin{aligned} sin(\theta) &= \frac{y}{\sqrt{R^2 + y^2}} \\ &= \frac{y}{R}\frac{1}{\sqrt{1 + (y/R)^2}} \\ &\approx \frac{y}{R}(1 - \frac{1}{2}\frac{y^2}{R^2}) \\ &\approx \frac{y}{R} \end{aligned} \tag{12a}$$

where we have used Taylor expansion of $\frac{1}{\sqrt{1+(y/R)^2}}$ and ignored all terms that are very small (terms that are higher than y/R). So, the y-component of the force of gravity on ball 2 is

$$F_{\text{ball 2}} sin(\theta) \approx F_{\text{ball 2}} \frac{y}{R} = \frac{GMmy}{R^3} \tag{13}$$

The y-component of gravity force that the moon exerts on ball 1 is zero. So if we say that the negative y-axis points downwards (towards the bottom of the page), then we have

$$F_{\text{ball 1, y}} - F_{\text{ball 2, y}} = \frac{GMmy}{R^3} \tag{14}$$

(**g**) Consider the setups shown in Figure 2(A) and (B). In Figure 2(A), consider the side of Earth that's closer to the moon. Here, the ocean bulges outwards, towards the moon as drawn in the picture. To understand this, we can consider "ball 1" to be the surface of the ocean and "ball 2" to be the ocean floor (solid Earth). Then the surface of the ocean, being closer to the moon than the ocean floor, will have higher gravitational pull towards the moon than the ocean bottom. Here, we would observe a "high tide". In Figure 2(A), the side of the Earth that is furtherest away from the moon will also experience a high tide. Here, we let "ball 1" be the ocean bottom and "ball 2" to be the surface of the ocean. Then the ocean bottom is pulled stronger towards the moon than the ocean surface. As an observer standing on Earth, you would see the ocean surface "lag" behind and thus bulging outwards as a high tide. It's crucial to note here that the tide is a purely *relative* phenomenon. Nothing actually "pulling" the ocean surface away from the moon. We can use the same logic to explain the low tides in Figure 2(B).

Solution to problem 2.

(**a**) On a given satellite of mass m, the centripetal force points radially towards the center of the circle (center of the planet of mass M). The magnitude of the centripetal force $\vec{F_c}$ is

$$\begin{aligned} F_{\text{c}} &= \frac{GMm}{R^2} + \frac{Gm^2}{(2R)^2} \\ &= \frac{GMm}{R^2} + \frac{Gm^2}{4R^2} \end{aligned} \tag{15a}$$

This should also be equal to $\frac{mv^2}{R}$.

(**b**) The net force acting on the Earth is zero because it is pulled by each planet in radially *opposite* directions (i.e. one satellite pulls the Earth radially towards itself, while the other satellite (which is exactly on the opposite side of the circle) pulls the Earth towards itself (thus radially opposing the pull of the first satellite).

(**c**) From (a), we see two formulas for the centripetal force. We equate the two to get

$$\frac{mv^2}{R} = \frac{GMm}{R^2} + \frac{Gm^2}{4R^2}$$
$$\implies v^2 = \frac{G(4M+m)}{4R}$$
$$\implies v = \sqrt{\frac{G(4M+m)}{4R}} \tag{16a}$$

Then since the period T is the time taken to go around the circle of radius R once, we have $vT = 2\pi R$ and thus

$$\implies T = \frac{2\pi R}{v}$$
$$\implies T = 2\pi R\sqrt{\frac{4R}{G(4M+m)}}$$
$$\implies T = 4\pi\sqrt{\frac{R^3}{G(4M+m)}} \tag{17a}$$

(**d**) This orbit is unstable. To see this, note that if we push the Earth towards one of the satellites at some instant, the Earth will be closer to one satellite than the other. Then the satellite that the Earth is closer to, will exert more gravitational pull on the Earth than the other satellite would on the Earth. Thus the Earth will now experience a non-zero net force. As a result, the Earth will start to fall into the satellite that it's closer to while that satellite is also moving. The other satellite will also be affected because the Earth now moves. The motion of all three bodies will be quite complicated. But we can at least see that this circular orbit is disrupted and that the Earth will likely (and it will) eventually collide with one of the satellites (and eventually the other satellite as well). So this circular orbit that we are describing in this problem is a highly engineered situation that can fall apart if you make even a small perturbation to the system. This is what we call an "unstable" system.

(**e**) The two satellites (one of mass m and the other of mass $m+\delta m$ cannot be orbiting in the same circle any more. This is because the Earth will experience a non-zero net force on it due to the satellite with mass $m + \delta m$ pulling on it more strongly than the other satellite with mass m if the two satellites are the same radial distance away from the Earth. Thus the Earth will not stand still. Therefore a circular orbit with both satellites in the same circle moving at the same speed cannot exist. We can ask if it's possible for each satellite to occupy different circles (but both circles with the Earth at the center). But this is a complicated question and we will not consider this here.

Solution to problem 3.

(**a**) You do no work (i.e. work = 0). Because there is nothing in space (no other object) that exerts a force on the particle of mass m_1 (i.e. there's nothing to work against).

(**b**) Gravitational pull of m_1 does positive work on the particle of mass m_2 because the gravitational force on m_2 is in the same direction as it's displacement (both the force and displacement vectors point towards m_1's location). The work that m_1's gravitational pull (not your hand) does on m_2 is equal to $-\Delta U$ where U is the gravitational potential energy of m_2. Thus

$$\begin{aligned} W_{\text{done by gravity}} &= -\Delta U \\ &= \frac{Gm_1m_2}{r_{12}} \end{aligned}$$

Important to remember: the work that *you* do against gravity *increases* the gravitational potential energy (i.e. you put in energy that is stored as the potential energy). The work that gravity does to pull the particles inwards *decreases* the potential energy (i.e. gravity has taken out energy from the stored potential energy to bring the particles inwards).

(**c**) Let's denote the work that we calculated in (a) by W_0 and the work that we calculated in (b) by W_{12}. Then in bringing the third particle of mass m_3, the gravitational pull of m_1 on m_3 and the gravitational pull of m_2 on m_3 both do work in moving the particle of mass m_3 from infinity to its new position (this new position is at distance r_{12} from m_1 particle and at distance r_{23} from the m_2 particle). Let W_{13} be the work that the gravitational attraction between m_1 and m_3 does in bringing m_3 inwards and let W_{23} be the work that the gravitational attraction between m_2 and m_3 does in bringing m_3 inwards. Using the same reasoning as in (b), we have

$$W_{13} = \frac{Gm_1m_3}{r_{13}} \qquad W_{23} = \frac{Gm_2m_3}{r_{23}} \tag{18}$$

Then the total work done by gravity to assemble the system of three particles is

$$\begin{aligned} W_{\text{done by gravity}} &= W_0 + W_{12} + W_{13} + W_{23} \\ &= 0 + \frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}} \end{aligned}$$

Note that the change in the gravitational potential energy in going from the empty space (initial configuration in (a)) to this final configuration of the three particles is $\Delta U = -W_{\text{done by gravity}}$, so

$$\Delta U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \tag{19}$$

And since initially (in (a)), all three particles are infinitely apart from each other, the initial gravitational potential energy is zero. So $\Delta U = U_{final} - 0 = U_{\text{final}}$, where U is the gravitational energy of the final configuration of the three particles:

$$U_{\text{final}} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \tag{19}$$

(**d**) The total work W_{you} that *you* do in bringing each particle out to infinity, starting from the final configuration of the three particles in (c), is equal to the change in the gravitational potential energy ΔU in going from the initial configuration (all particles together) to the final configuration (all particles out in infinity). We calculated the change in potential energy in the reverse process in (c). So our ΔU must be -1 times the change in potential energy that we calculated in (c). So

$$\begin{aligned} W_{\text{done by you}} &= \Delta U \\ &= \frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}} \end{aligned}$$

As a sanity check: Note that the work done by you is positive (and thus you put in energy in the form of gravitational potential energy). This makes sense since you are doing work *against* gravity. So you have to put in energy (which increases the potential energy).

(**e**) Look at the U_{final} that we calculated in (c). With the four particles in their final configuration (Figure 4 on the problem sheet), we need to apply the same reasoning that we used for calculating the U_{final} in (c). The work that gravity does to assemble all four particles must be the work that gravity does to assemble the three particles (calculated in (c)) plus the work done to bring in the fourth particle (of mass m_4). Let W_{14} be the work that the gravitational pull of m_1 does to bring in m_4, W_{24} be the work that the gravitational pull of m_2 does to bring in m_4, and W_{34} be the work that the gravitational pull of m_3 does to bring in m_4. Then the total work done by gravity to assemble all four particles (bringing all four particles from out in infinity to the final configuration of the four particles) is

$$\begin{aligned} W_{\text{done by gravity}} &= W_0 + W_{12} + W_{13} + W_{23} + W_{14} + W_{24} + W_{34} \\ &= 0 + \frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}} + \frac{Gm_1m_4}{r_{14}} + \frac{Gm_2m_4}{r_{24}} + \frac{Gm_3m_4}{r_{34}} \end{aligned}$$

Now, gravity takes energy out from potential energy to do this work. So the change in the gravitational potential energy is $\Delta U = -W_{\text{done by gravity}}$. And since the initial potential energy (when everyone is at infinity) is zero, we have $\Delta U = U_{final}$. So

$$\begin{aligned} U_{final} &= -W_{\text{done by gravity}} \\ &= -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_4}{r_{14}} - \frac{Gm_2m_4}{r_{24}} - \frac{Gm_3m_4}{r_{34}} \end{aligned}$$

This is the answer and it's fine to say that we're done. But we can actually write above equation in a compact (and thus nicer) form. Note that our reasoning above was one that involved counting pairs. We make sure that we account for every pair of particles, and then write down the energy for each pair. Now, note that each pair is counted *once*.

But suppose we count each pair *twice.* In other words, say we count the gravitational attraction that m_1 has on m_2 as one pair, and we also count the gravitational attraction that m_2 has on m_1 as another pair (but in fact, they are the same). So if we add the gravitational energy of these two pairs, then it's the double the gravitational energy of one of them. Say we do this "double" counting for all pairs, then above equation for U_{final} becomes

$$U_{final} = -\frac{1}{2}\sum_{i=1}^{4}\sum_{\substack{j=1\\ j\neq i}}^{4}\frac{Gm_im_j}{r_{ij}} \tag{22}$$

Another way to check this is by writing the above sum explicitly, term by term. Then you will get the same expression for U_{final} as before.

(**f**) The total work that *you* do to bring all the particles out to infinity, starting from all four particles together as a polygon, is -1 times the change in gravitational potential energy you computed in (e). So it is

$$\begin{aligned} W_{\text{done by you}} &= -U_{final} \\ &= \frac{1}{2}\sum_{i=1}^{4}\sum_{\substack{j=1\\ j\neq i}}^{4}\frac{Gm_im_j}{r_{ij}} \end{aligned}$$

Solution to problem 4.

(**a**) Let's break the problem into the x-component and y-component. That is, we first find the x-coordinate of the center of mass and then find the y-coordinate of the center of mass separately (you can do this the other way around too). (x, y) = (0, 0) is the center of this hybrid sphere. We want to find the position (x_{cm}, y_{cm}) of the center of mass. Note that we can say, without calculations, that $x_{cm} = 0$. Why? Well, suppose it wasn't. Say $x_{cm} = 3$. Then the center of mass is to the right of the center of the sphere. But the sphere looks the same on the left side of the y-axis as it does on the right side of the y-axis (if you draw the y-axis, which is a vertical line that goes through the center (0, 0), then you see that the right side of the y-axis is a mirror image of the left side of the y-axis). So x_{cm} cannot be equal to 3 (because there's nothing special about the right side of the y-axis). So $x_{cm} = 0$ is the only position where it makes sense.

Calculating the y_{cm} involves actual calculation and it's actually complicated (involves integrals). Let's not do this.

(**b**) No calculation necessary here because the hollow sphere is still a symmetric object. The center of mass is at $(x_{cm},\, y_{cm}) = (0,\, 0)$ by symmetry.

(**c**) Let $(x_{cm},\, y_{cm})$ be the position of the center of mass of the sphere with the hole carved out of it (a "hollow sphere"). Let's first calculate the x_{cm}. We must have

$$Mx_{\text{full}} = m_{\text{hole}}x_{hole} + m_{\text{hollow}}x_{cm} \tag{23}$$

where x_{full} is the x-position of the full sphere's center of mass, m_{hole} and x_{hole} are the mass and the x-component of the sphere that was taken out to create the hole respectively, and m_{hollow} is the mass of the hollow sphere (the full sphere minus the hole). Now, we know that $x_{full} = 0$ and $x_{hole} = R/2$. We can calculate m_{hole} and m_{hollow}:

$$m_{\text{hole}} = \frac{M}{(4/3)\pi R^3}(4/3)\pi r^3 \qquad m_{\text{hollow}} = M - m_{\text{hole}} \tag{23}$$

Doing the above calculation, we get

$$m_{\text{hole}} = M\frac{r^3}{R^3} \qquad m_{\text{hollow}} = M(1 - \frac{r^3}{R^3}) \tag{23}$$

Plugging these into the first equation, we get

$$M \cdot 0 = M\frac{r^3}{R^3}\frac{R}{2} + M(1 - \frac{r^3}{R^3})x_{cm} \tag{23}$$

Solving for x_{cm}, we get

$$\begin{aligned} x_{cm} &= \frac{r^3}{2R^2} \cdot \frac{R^3}{r^3 - R^3} \\ \implies &= \frac{R}{2} \cdot \frac{r^3}{r^3 - R^3} \end{aligned}$$

As for the y_{cm}, we see that the image above the x-axis is exactly the same as the image below the x-axis (the two are mirror images of each other). Thus $y_{cm} = 0$. Putting everything together, we have

$$(x_{cm}, y_{cm}) = (\frac{R}{2} \cdot \frac{r^3}{r^3 - R^3}, 0) \tag{24}$$

Solution to problem 5

This is a straight forward calculation. Keep the total kinetic energy constant. And also keep the total momentum also constant over time.

Solution to problem 6

The key here is that the total momentum of the system (system = person + boat) always remains the same (remains at zero since nothing is moving initially). This is because there is no net external force on the system (i.e. nothing outside the system exerts a net force on the system). Gravity indeed acts on the system and gravity is an external force (because the Earth that exerts the gravity is not part of our system). But the gravitational force is cancelled out by the normal force that the water exerts on the system. So there is no *net* external force. In this case, we know that the total momentum remains unchanged (remains at zero in our case) and thus the center of mass of the system also remains stationary all all times. So if we calculate the position of the center of mass before the person starts to walk and then calculate it after the person stops. Let the initial position of the center of mass be $x = 0$ (this is a one-dimensional problem so no y here). If we say that the front of the boat is located on the positive side of the x-axis, we have $x_{person} = L/2$ being the person's initial position. So initially, the center of mass is at $x_{\text{cm, initial}}$, where

$$\begin{aligned} x_{\text{cm, initial}} &= \frac{mL/2 + M \cdot 0}{m + M} \\ &= \frac{mL}{2(m + M)} \end{aligned}$$

This is before the person starts to walk. As the person walks, the boat will move too. But the center of mass of the whole system remains fixed at the same location. When the person stops, the boat stops moving, and the center of mass of the whole system again remains fixed at the same location. When the person stops, the center of the boat is at position x_{boat} and the girl is at $x = x_{boat} - L/2$ (because the girl will be to the left of the center of the boat by distance L/2). And the center of mass is at $x_{cm,initial}$. So we have

$$\begin{aligned} x_{cm,initial} &= \frac{Mx_{boat} + m(x_{boat} - L/2)}{m + M} \\ \Longrightarrow \quad \frac{mL}{2(m + M)} &= \frac{Mx_{boat} + m(x_{boat} - L/2)}{m + M} \end{aligned}$$

Solving for x_{boat} gives us

$$x_{boat} = \frac{mL}{M + m} \tag{26}$$

This is the location of the boat's center relative to the initial position (which was zero) of the boat's center.

Solution to problem 7

(**a**) As the sand is being added to the moving conveyor belt, if you don't add any force, the conveyor belt will slow down and eventually stop. The way to see this is that if the conveyor belt has mass M, and is initially moving at velocity v (no arrow on top of the v here because it's one-dimension (so v can be positive or negative and that takes care of the direction)). The the initial momentum of the conveyor belt is Mv. Just when the sand grain hits the conveyor belt and lands on it, the sand exerts a force on the conveyor belt. By Newton's 3rd law, the conveyor belt exerts a force on the sand that is equal in magnitude and opposite in direction. This pair of internal forces cancel out. Importantly, there is no external force on this system. So the total momentum must be conserved in this collision between the belt and the sand. If the sand has mass m, then the total momentum must be $(m + M)u$, where u is the new velocity of the belt with the sand on top of it (the sand and the belt are now moving together at velocity u). To have the total momentum of the system being conserved, we must have $(m+M)u = Mv$. This can only happen if $u < v$. So without you (or some external body) exerting some external force on the system, the belt will slow down and eventually stop. To keep the belt moving at a constant velocity v, we must exert a force $\vec{F}_{you}$ such that

$$\vec{F}_{you} = -\frac{d\vec{p}}{dt} \tag{26}$$

where $\vec{p}$ is the total momentum of the system (system = sand + belt). The negative sign in front of the derivative is saying that you have to apply a force that counteracts the loss of momentum over time (i.e. dp/dt ¡ 0 without your force). Now if we let $M(t)$ be the total mass of the system at time t and the system (=sand already on the belt + belt) is always moving at a constant velocity v because you're applying a force F on it, then the system's momentum at time t is $M(t)v$. And we must have

$$\begin{aligned} F_{\text{you}} &= \frac{d}{dt}(M(t)v) \\ &= \frac{dM(t)}{dt}v \\ &= Rv \end{aligned}$$

which is a constant. So you have to apply a constant force $F = Rv$ at all times to keep the belt moving at the constant velocity v.

(**b**) Assume that each sand grain is just gently placed on the moving belt (so the initial grain has zero kinetic energy). After each sand grain lands on the moving belt, they move at speed v. So at time t, the system's total kinetic energy (= kinetic energy of belt + kinetic energy of sand on the belt) is

$$KE = \frac{M(t)v^2}{2} \tag{27}$$

where $M(t)$ is the total mass of the system at time t (as in (a)). So

$$\frac{d(KE)}{dt} = \frac{v^2}{2}\frac{dM(t)}{dt} \tag{27}$$

because v is constant over time. We know that $dM/dt = R$ so

$$\frac{d(KE)}{dt} = \frac{Rv^2}{2} \tag{27}$$

This is rate of kineetic energy gain by the sand particles (not the belt because the belt's mass doesn't change over time and the belt's speed always remains constant).

(**c**) You constantly apply a force $F = Rv$. And power P is $P = Fv$. So the power (= work you do per unit time) is Rv^2. Note that if you don't remember that $P = Fv$, it's okay. The only thing you need to remember is that power is, by definition, rate of change of energy (work) per unit time. So

$$\begin{aligned}
P &= \frac{dW}{dt} \\
&= \frac{d(F dx)}{dt} \\
&= F\frac{dx}{dt} \\
&= Fv \\
&= Rv^2
\end{aligned}$$

where dx is the infinitesimal distance that the system (=belt + sand on top of it) travels in time interval dt.

(**d**) In (b), we found that per unit time, the kinetic energy of the system increases by $Rv^2/2$. In (c) we find that you put in energy (you do work) into the system per unit time by an amount Rv^2. So you put in more energy than the system gets per unit time. The difference, $Rv^2/2$ must be lost to heat.