NB1140: Physics 1A - Classical mechanics and Thermodynamics
Solution set 4 - Rotational motion, torque, angular momenta, oscillations, and waves
Week 7: 9 - 13 January 2017

Solution to problem 1.

(**a**) Rotational inertia I_k of the cylindrical kroket about its axis of symmetry. The cylinder is a solid cylinder with a uniform mass density inside it. The mass density ρ is total mass M divided by the cylinder's volume. Thus it is

$$\rho = \frac{M}{\pi R^2 L} \tag{1}$$

We can consider the solid cylinder to be made of layers of concentric hollow cylinders. Each hollow cylinder has length L and an infinitesimal thickness dr. Let r_i be the radius of the concentric i-th cylinder. Then I_k is the sum of the rotational inertia of each concentric cylinder:

$$I_k = \sum_i (\rho 2\pi r_i L dr) r_i^2 \tag{2}$$

The total mass of each cylindrical slab is $(\rho 2\pi r_i L dr)$. Above formula is nothing other than the familiar formula for rotational inertia, $m_i r_i^2$. Seeing the dr in the summation, we note that we can write above as an integral with r being the variable to integrate over from $r = 0$ to $r = R$:

$$\begin{aligned} I_k &= \int_0^R \rho 2\pi L r^3 dr \\ &= 2\rho\pi L \frac{r^4}{4}\Big|_0^R \\ &= \frac{\rho\pi L R^4}{2} \\ &= \frac{M\pi L R^4}{2\pi R^2 L} \\ &= \frac{MR^2}{2} \end{aligned} \tag{3a}$$

Thus the rotational inertia of the kroket is $\frac{MR^2}{2}$.

(**b**) We calculate the rotational inertia I_b of the spherical bitterball with a method similar to the one we used in (a). We break up the spherical bitterball into concentric layers of spherical slabs. Each spherical slab has a thickness dr. The radius of the i-th spherical slab is r_i. I_b is the sum of the rotational inertia of individual spherical slabs. The mass M is uniformly distributed throughout the spherical bitterball. So we have the uniform mass density ρ to be

$$\rho = \frac{3M}{4\pi R^3} \tag{4}$$

The bitterball's rotational inertia is thus

$$\begin{aligned} I_b &= \sum_i \rho 4\pi r_i^2 dr r_i^2 \\ &= \int_0^R \rho 4\pi r^4 dr \\ &= 4\rho\pi \frac{R^5}{5} \\ &= \frac{3MR^2}{5} \end{aligned} \tag{5a}$$

(**c**) Since there is no rolling motion at all and both objects slide down the incline, the shape of the object doesn't matter at all. This is the typical "block sliding down an inclined plane" problem. The kroket and the bitterball have the same mass M. Thus they both have the same acceleration $a = gsin(\theta)$. Thus both reach the bottom of the inclined plane at the same time.

(**d**) Both roll down the inclined ramp without slipping. Both are released at the same time from the top of ramp. The torque τ about the rotational axis of each object is

$$\tau = \vec{r} \times \vec{F_f} \tag{6}$$

where $\vec{F_f}$ is the frictional force between the point of the contact of the rolling object and the ramp and $\vec{r}$ is the position vector that emanates from the center of the rolling object and terminates at the contact point. The magnitude of the torque vector is

$$|\tau| = RF_f \tag{7}$$

Note that $I\vec{\alpha} = \vec{\tau}$, where $\vec{\alpha}$ is the angular acceleration. Thus we have

$$\alpha = \frac{RF_f}{I} \tag{8}$$

We removed the arrows on top of α and τ and in doing so, we are saying that both can be either positive or negative numbers (their signs determine the direction of each vector). Rolling without slipping means that if the wheel rotates once, the center of the wheel moves a distance equal to the circumference of the wheel. This means that $R\alpha = a$, where a is the acceleration of the wheel. Thus above equation becomes

$$a = \frac{R^2 F_f}{I} \tag{9}$$

By Newton's 2nd law, we have (for both objects):

$$\begin{aligned} Ma &= F_{\text{total}} \\ &= Mgsin(\theta) - F_f \end{aligned} \tag{10a}$$

where we use "+" direction to be downwards along the ramp and "-" direction to be upwards along the ramp. From equation (9), we can solve for F_f:

$$F_f = \frac{Ia}{R^2} \tag{11}$$

Plugging this into equation (10), we get

$$Ma = Mgsin(\theta) - \frac{Ia}{R^2} \tag{12}$$

Solving for a, we get

$$\begin{aligned} a &= gsin(\theta) - \frac{Ia}{MR^2} \\ \implies \frac{a(MR^2 + I)}{MR^2} &= gsin(\theta) \\ \implies a &= \frac{MR^2 gsin(\theta)}{I + MR^2} \end{aligned} \tag{13a}$$

Note that the bitterball and the kroket have different values for I. But in both objects (and for any objects), we can express the rotational inertia as cMR^2, where c is some dimensionless number that depends on the shape and mass distribution of the object. So plugging this into above equation, we have

$$\begin{aligned} a &= \frac{MR^2 gsin(\theta)}{cMR^2 + MR^2} \\ &= \frac{gsin(\theta)}{1 + c} \end{aligned} \tag{14a}$$

From parts (a) and (b), we know that the bitterball has rotational inertia $I_b = \frac{3MR^2}{5}$ and the kroket has rotational inertia $I_k = \frac{MR^2}{2}$. Thus for the bitterball, $c = 3/5$ and for the kroket, $c = 1/2$. Plugging these into above equation, we have

$$a_b = \frac{5gsin(\theta)}{8} \qquad a_k = \frac{2gsin(\theta)}{3} \tag{15}$$

a_k is slightly larger than a_b. Thus the kroket arrives at the bottom of the incline before the bitterball.

Solution to problem 2.

(**a**) This problem appears on Quiz 3.

(**b**) The key in this problem is that the total angular momentum of the system (system = disc + insect) is conserved. This means that the total angular momentum of the system before the bug lands on the disc is the same as the total angular momentum of the system after the bug lands on the disc. Before landing on the disc, the total angular momentum of the system is

$$\begin{aligned}\vec{L}_{before} &= \vec{L}_{bug} + \vec{L}_{disc} \\ &= mvd\hat{z} + I\omega\hat{z} \\ &= (mvd + I\omega)\hat{z}\end{aligned} \tag{16a}$$

After the bug lands, the disc and the bug spin together with angular velocity ω_f. The total angular momentum of the system after the bug lands is

$$\begin{aligned}\vec{L}_{after} &= \vec{L'}_{bug} + \vec{L'}_{disc} \\ &= md^2\omega_f\hat{z} + I\omega_f\hat{z} \\ &= (md^2 + I)\omega_f\hat{z}\end{aligned} \tag{17a}$$

By conservation of total angular momentum, we have

$$\begin{aligned}(mvd + I\omega) &= (md^2 + I)\omega_f \\ \Longrightarrow \omega_f &= \frac{mvd + I\omega}{md^2 + I}\end{aligned} \tag{18a}$$

Thus to have no change in angular speed (i.e. $\omega_f = \omega$), we must have

$$\begin{aligned}md^2\omega &= mvd \\ \Longrightarrow \ v &= \omega d\end{aligned} \tag{19a}$$

(**c**) To double the angular speed (i.e. $\omega_f = 2\omega$), we must have

$$\begin{aligned}2md^2\omega + I\omega &= mvd \\ \Longrightarrow \ v &= 2\omega d + \frac{I}{md}\omega \\ \Longrightarrow \ &= (2d + \frac{I}{md})\omega\end{aligned} \tag{20a}$$

Solution to Problem 3

(**a**) Let I be the rotational inertia of the uniform disc of radius R. Let ω be the initial rotational speed ($\omega = 0.25$ rad /s). Initially, everyone was rotating together at angular speed ω. So the total angular momentum was initially

$$L_{initial} = (I + mR^2)\omega \tag{21}$$

After the cockroach walks halfway to the center of the disk, it's at a distance $R/2$ from the rotational axis (the center of the disc). The cockroach stops and the whole system spins together at angular velocity ω_f. The total angular momentum is conserved. So we have

$$\begin{aligned} L_{initial} &= L_{final} \\ (I + mR^2)\omega &= (I + m\frac{R^2}{4})\omega_f \\ \omega_f &= \frac{I + mR^2}{I + mR^2/4}\omega \end{aligned} \tag{22a}$$

For a uniform circular disc, we have $I = MR^2/2$. We also have $M = 10m$. Thus we have $I = 5mR^2$ and

$$\omega_f = \frac{24}{21}\omega \tag{23}$$

(**b**)

$$K_o = \frac{I + mR^2}{2}\omega^2 \tag{24}$$

$$K = \frac{I + mR^2/4}{2}\omega_f^2 \tag{25}$$

Thus

$$\begin{aligned} \frac{K}{K_o} &= \frac{\omega_f^2}{\omega^2}\frac{I + mR^2/4}{I + mR^2} \\ &= (\frac{24}{21})^2\frac{21}{24} \\ &= \frac{24}{21} \end{aligned} \tag{26a}$$

(**c**) The cockroach has done work on the system (the energy comes from the chemically stored energy inside the cockroach (e.g. ATP –¿ ADP)). This has increased the total system's kinetic energy (while the potential energy that was chemically stored in the cockroach disappears).

Solution to Problem 4

This problem was solved in class.

Solution to Problem 5

a Substituting $x(t)$ into the differential equation immediately gives $m\lambda^2 + \gamma\lambda + k = 0$. Solving this quadratic equation gives two solutions, $\lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$, except for the special case $\gamma = 2\sqrt{km}$, where $\lambda = -\gamma/2m$.

b The argument in the root is negative, so

$$x(t) = e^{-\gamma t/2m}\left(Ae^{+i\sqrt{4mk-\gamma^2}t/2m} + Be^{-i\sqrt{4mk-\gamma^2}t/2m}\right),$$
$$x(t) = e^{-\gamma t/2m}\left((A+B)\cos(\sqrt{4mk-\gamma^2}t/2m) + (Ai - Bi)\sin(\sqrt{4mk-\gamma^2}t/2m)\right).$$

Because $x(t=0)=0$ we need the first term to vanish, so $A=-B$. Furthermore we have $v(t=0)=v_0$, so $Ai = mv_0/\sqrt{4mk-\gamma^2}$, and we find

$$x(t) = \frac{2mv_0}{\sqrt{4mk-\gamma^2}}e^{-\gamma t/2m}\sin(\sqrt{4mk-\gamma^2}t/2m).$$

c

$$x(t) = e^{-\gamma t/2m}\left(Ae^{+\sqrt{\gamma^2-4mk}t/2m} + Be^{-\sqrt{\gamma^2-4mk}t/2m}\right)$$

The boundary condition $x(t=0)=0$ sets $A=-B$. We also have $v(t=0)=v_0$, from which we find $A = mv_0/\sqrt{\gamma^2-4mk}$, and:

$$x(t) = \frac{mv_0}{\sqrt{\gamma^2-4mk}}e^{-\gamma t/2m}\left(e^{+\sqrt{\gamma^2-4mk}t/2m} - e^{-\sqrt{\gamma^2-4mk}t/2m}\right).$$

d $\lambda = -\gamma/2m$. At $t=0$ we have $x(t)=0$, so $A=0$ and the other boundary condition gives $B=v_0$. Therefore $x(t) = v_0 t e^{-\frac{\gamma t}{2m}}$

Solution to Problem 6

In class.