Problem 1

1.a)

N particles: 3N positions + 3N velocities

$$\Omega = (\#\text{ of position states}) \times (\#\text{ of velocity states})$$

The number of position states of one particle is

$$\frac{V_{tot}}{V_p}$$

where V_p stands for the volume of the particle and V_{tot} for the total volume of the system. Since there are N identical particles the total number of position states is given by

$$\# \text{ of position states} = \left(\frac{V_{tot}}{V_p}\right)^N$$

To calculate the number of velocity states we note that the energy of the system remains constant

$$E = \frac{1}{2}m\sum_{i=1}^{3N} v_i^2 = constant$$

rearranging the equation we get

$$\sum_{i=1}^{3N} v_i^2 = \frac{2E}{m}$$

which corresponds to the equation of a $(3N-1)-$dimensional hyper-sphere with radius $\sqrt{2E/m}$ in a $3N-$dimensional space with coordinates $\{v_{1x}, v_{1y}, ..., v_{Nz}\}$. So the set of all possible velocities contains all the vectors that start in the origin and end on the hyper-sphere.

$$\# \text{ of velocity states} = \frac{A}{c_v}$$

here, A represents the area of the hyper-sphere and c_v is just a constant with the same dimensions as A. Since the area of a 1-dimensional sphere (circle) with radius r is $2\pi r$ and the area of a 2-dimensional sphere is $4\pi r^2$, we see that the area of a $(3N-1)-$dimensional sphere will be given by $c_a r^{3N-1}$; where c_a is a constant. In our problem, the radius of the sphere is $\sqrt{2E/m}$, so the number of velocity states is

$$\# \text{ of velocity states} = \frac{c_a\left(\frac{2E}{m}\right)^{\frac{3N-1}{2}}}{c_v}$$

Therefore

$$\Omega = \left(\frac{c_a}{V_p^N c_v}\right)\left(V^N\right)\left(\frac{2E}{m}\right)^{\frac{3N-1}{2}}$$

Entropy is simply given by $S = k_B \ln \Omega$

$$S = k_B N \ln V + k_B \frac{3N}{2} \ln\left(\frac{2E}{m}\right) + k_B \ln(constant)$$

where we have used the fact that N is large to make the approximation $\frac{3N-1}{2} \approx \frac{3N}{2}$.

1.b)

if $volume = V$

$$S_1 = k_B N \ln V + k_B \frac{3N}{2} \ln\left(\frac{2E}{m}\right) + k_B \ln(constant)$$

if $volume = 2V$

$$S_2 = k_B N \ln 2V + k_B \frac{3N}{2} \ln\left(\frac{2E}{m}\right) + k_B \ln(constant)$$

so

$$\Delta S = S_2 - S_1 = k_B N(\ln 2V - \ln V) = k_B N \ln\left(\frac{2V}{V}\right) = k_B N \ln 2 > 0$$

Problem 2

We know that it is a monoatomic and ideal gas. So $pV = nRT$. We also know that $V_1 = 7V_0$.

2.a)

Isothermal, so $T_0 = T_1 = T$. So for the p_1 we can do:

$$\begin{aligned} p_1 &= \frac{nRT_1}{V_1} = \frac{nRT}{7v_0} && (16) \\ & && (17) \end{aligned}$$

For the work we can fill in the ideal gas law in the formula for work.

$$\begin{aligned} W &= -\int_{V_0}^{V_1} p\,\mathrm{d}V && (18) \\ W &= -\int_{V_0}^{V_1} \frac{nRT}{V}\,\mathrm{d}V && (19) \\ W &= -nRT\,ln(7) && (20) \end{aligned}$$

To calculate the heat, Q, we use the formula $\Delta U = Q + W$. The change in internal energy is zero, because it only depends on temperature. This gives for the heat:

$$Q = -W = nRT\,ln(7) \quad (21)$$

2.b)

Isobaric expansion. So the pressure is constant, $p_0 = p_1$.:

$$p_1 = \frac{nRT_0}{V_0} \quad (22)$$

To calculate the final temperature we use the ideal gas law:

$$\begin{aligned} T_1 &= \frac{p_1 V_1}{nR} = \frac{7p_0 v_0}{nR} && (23) \\ T_1 &= 7T_0 && (24) \end{aligned}$$

For the work we use again the same formula. However, now the pressure is constant, so we can take it out of the integral.

$$\begin{aligned} W &= -\int_{V_0}^{V_1} p\,\mathrm{d}V && (25) \\ W &= -p\int_{V_0}^{V_1} \mathrm{d}V && (26) \\ W &= -6p_0V_0 = -6nRT_0 \end{aligned}$$

In this case the internal energy of the system changes because the temperature changes. Therefor, we need te calculate the internal energy change before we can calculate the heat. We know that a monoatomic gas has 3 degrees of freedom. For this reason we now that the internal energy is given by $U = \frac{3}{2}nRT$. Therefor, the internal energy difference is:

$$\Delta U = \frac{3}{2}nR(T_1 - T_0) \quad (28)$$
$$\Delta U = 9nRT_0 \quad (29)$$

We fill this in in the equation $\Delta U = Q + W$ and we get:

$$Q = 9nRT_0 - -6nRT_0 \quad (30)$$
$$Q = 15nRT_0 \quad (31)$$

2.c)

In this case the expansion is adiabatic. So, the heat is zero, $Q = 0$. We also know that $pV^\gamma = constant$ and $TV^{\gamma-1}$. We will use this to find P_1 and T_1. Before we can do that, we need to know gamma. We know $\gamma = \frac{C_p}{C_v}$.

$$C_v = \frac{1}{n}\frac{\Delta U}{\Delta T} \quad (32)$$
$$C_v = \frac{3}{2}nR \quad (33)$$
$$C_p = C_v + R = \frac{5}{2}R \quad (34)$$
$$\gamma = \frac{C_p}{C_v} = \frac{5}{3} \quad (35)$$

Now, we can find p_1 and T_1

$$p_1 = \frac{p_0 V_0^\gamma}{V_1^\gamma} \quad (36)$$
$$p_1 = \frac{p_0}{7^{\frac{5}{3}}} = 7^{-\frac{5}{3}}\frac{nRT_0}{V_0} \quad (37)$$
$$T_1 = \frac{T_0 V_0^{\gamma-1}}{V_1^{\gamma-1}} \quad (38)$$
$$T_1 = \frac{T_0}{7^{\frac{2}{3}}} \quad (39)$$

In problem 1a we showed that in an adiabatic expansion the work is given by: $W = \frac{p_2V_2 - p_1V_1}{\gamma-1}$. So, our work will be:

$$W = \frac{7^{-\frac{5}{3}}\frac{nRT_0}{V_0}7V_0 - \frac{nRT_0}{V_0}V_0}{\frac{2}{3}} \quad (40)$$
$$W = \frac{3}{2}nRT_0(7^{-\frac{2}{3}} - 1) \quad (41)$$

Problem 3

3.a)

We have defined efficiency as

$$\eta = \left|\frac{\text{Work generated}}{\text{Heat supplied by the heat source}}\right|$$

We see from the Figure that Q_1 corresponds to the energy supplied by the heat source and Q_2 corresponds to the energy rejected to the heat sink. Also, since the internal energy U is a function of state, the total change of energy ΔU in a whole cycle is zero, and hence from the First Law of Thermodynamics ($\Delta U = Q + W$) we see that

$$|W| = |Q| = |Q_1| - |Q_2|$$

Therefore,

$$\eta = \left|\frac{W}{Q_1}\right| = \frac{|Q_1| - |Q_2|}{|Q_1|} = 1 - \frac{|Q_2|}{|Q_1|}$$

3.b)

From Problem 1.b we see that for an adiabatic process connecting two points, A and B, in the PV diagram we have

$$W_{A\to B} = \frac{P_B V_B - P_A V_A}{\gamma - 1}$$

3.b.i) Adiabatic compression from 1 to 2

$$W_{1\to 2} = \frac{P_2 V_2 - P_1 V_1}{\gamma - 1}$$

3.b.ii) Adiabatic expansion from 3 to 4

$$W_{3\to 4} = \frac{P_4 V_4 - P_3 V_3}{\gamma - 1} = \frac{P_4 V_1 - P_3 V_2}{\gamma - 1}$$

3.c)

From Problem 3.a we have

$$\eta = 1 - \frac{|Q_2|}{|Q_1|}$$

From the First Law of Thermodynamics ($dU = dQ + dW$), the relation $dU = C_v dT$ (which implies $\Delta U = C_v \Delta T$ for constant C_v) and the fact that the process $(2 \to 3)$ is isochoric we have

$$\Delta U_{2\to 3} = C_v \Delta T_{2\to 3}$$

and

$$\Delta U_{2\to 3} = Q_1 - \int_{V_2}^{V_3=V_2} P\,dV = Q_1$$

And therefore

$$Q_1 = C_v(T_3 - T_2)$$

Following an identical reasoning for process $(4 \to 1)$ we get

$$Q_2 = C_v(T_1 - T_4)$$

So the efficiency can be written as

$$\eta = 1 - \frac{|T_3 - T_2|}{|T_1 - T_4|} = 1 - \frac{T_3 - T_2}{T_4 - T_1}$$

Rearranging the terms we obtain

$$\eta = 1 - \frac{T_2}{T_1}\left[\frac{T_3/T_2 - 1}{T_4/T_1 - 1}\right] \tag{42}$$

From Problem 1.*a* we know that for an adiabatic process

$$TV^{\gamma-1} = \text{constant}$$

and therefore

$$T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1} \tag{43}$$

$$T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1} \tag{44}$$

Taking the ratio of Eq. 43 to Eq. 44

$$\frac{T_3 V_3^{\gamma-1}}{T_2 V_2^{\gamma-1}} = \frac{T_4 V_4^{\gamma-1}}{T_1 V_1^{\gamma-1}} \tag{45}$$

and noting that $V_2 = V_3$ and $V_1 = V_4$ we see that the volumes in Eq. 45 cancel out, and therefore

$$\frac{T_3}{T_2} = \frac{T_4}{T_1} \tag{46}$$

Using Eq. 46 we can see that the term in brackets in Eq. 42 is equal to 1, and hence

$$\eta = 1 - \frac{T_2}{T_1} \tag{47}$$

Now we can rearrange Eq. 44 to get

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

and substitute that in Eq. 47 to obtain

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Finally, since $r = V_1/V_2$

$$\eta = 1 - \left(\frac{1}{r}\right)^{\gamma-1} = 1 - \left(r^{-1}\right)^{\gamma-1} = 1 - r^{1-\gamma}$$