NB1140: Physics 1A
Lecture 1: Kinematics - Description of objects' motions
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Kinematics concerns describing how objects (atoms, blocks, cells, humans) move *without* explaining *why* the objects move the way they do. In other words, we describe the position, velocity, and acceleration of the object without going deeper into what's causing the object to have such position, velocity, and acceleration (i.e., without going into what forces are acting on the object). In the next lecture, we will study **dynamics**, which includes kinematics in addition to studying how forces acting on the objects cause the motion.

1 1-dimensional motion: Motion along a line

One-dimensional motion of an object means that the position of the particle at all times is confined along a line.

- $x(t)$ = position of object at time t
- $v(t)$ = velocity of object at time t
- $a(t)$ = acceleration of object at time t

When do you put an arrow above the x, v, and a to write them as $\vec{x}$, $\vec{v}$, $\vec{a}$? You put an arrow if you have a vector (=number with a direction) instead of a simple number. In 1-dimension, there are only two directions: left and right on the number line (i.e., towards the negative direction or towards the positive direction). That's why for objects whose motions are confined to 1-dimension, we don't need to put an arrow above the x, v, and a. But in 2-dimensions and 3-dimensions, there are infinite number of directions, not just two. So in 2-dimensions and 3-dimensions, putting an arrow above the $\vec{x}$, $\vec{v}$, and $\vec{a}$ makes a big difference in the meaning of those variables.

In one dimension, we only need to know the following on how $x(t)$, $v(t)$, and $a(t)$ are related to each other:

Relationships among x(t), v(t), and a(t):

$$v(t) = \frac{dx}{dt} \tag{1}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2}$$

And since integral is the inverse operation of differentiation, taking the integral of above equations gives us the following:

"Inverse relationships" among x(t), v(t), and a(t):

$$x(t) - x(t_o) = \int_{t_o}^{t} v(\tau)d\tau \tag{3}$$

$$v(t) - v(t_o) = \int_{t_o}^{t} a(\tau)d\tau \tag{4}$$

You might remember from your analysis (calculus) class that the τ is what we call a "dummy variable" - it's only purpose is to be inside the integral (Note that we cannot use t and dt inside the integral because that t is a specific time that we're interested in (and is thus used outside the integral: For example in equation (3) we have $x(t)$ - position at a specific time t.

Example 1: Constant acceleration (also called "Uniform acceleration") in 1-dimension:
Say an object is accelerating at a constant rate A. Determine the object's velocity $v(t)$ and position $x(t)$ for any time t.

Solution:

$$v(t) - v(t_o) = \int_{t_o}^{t} Adt = A \cdot (t - t_o) \tag{5a}$$

$$\begin{aligned} x(t) &= x(t_o) + \int_{t_o}^{t} v(\tau)d\tau \\ &= x(t_o) + \int_{t_o}^{t} [v(t_o) + A \cdot (\tau - t_o)]d\tau \\ &= x(t_o) + v(t_o) \cdot (t - t_o) - At_o \cdot (t - t_o) + \frac{At^2}{2} - \frac{At_o^2}{2} \end{aligned} \tag{5b}$$

Equation (5b) becomes simpler if just say that we will set t_o to be zero. That is, we decide to start counting time (i.e., click on the "start" button on our timer). Then equation (5b) becomes:

$$x(t) = x(0) + v(0) \cdot t + \frac{At^2}{2} \tag{6}$$

You probably saw this equation in your high school class. Now you know how to derive it with calculus (derivatives and integrals).

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2 2-dimensional and 3-dimensional motions: Moving in a plane (2-dimensions) and space (3-dimensions)

Basically, 2-dimensional motion means that we simultaneously do 2 times (1-dimensional motion) and 3-dimensional motion means that we simultaneously do 3 times (1-dimensional motion). So we can use the knowledge from above section and separately apply to each dimension.

In two dimensions, an object moves in a plane (e.g., xy-plane, yz-plane, xz-plane). In other words, if an object is living on a piece of paper, it moves on the surface of the paper and never leaves that surface. Here we need to use the "arrow" above the position, velocity, and acceleration terms because they are **vectors** now:

Relationships among $\vec{r}$(t), $\vec{v}$(t), and $\vec{a}$(t): 2D motion:

$$\vec{r}(t) = (x(t), y(t)) \tag{7}$$

$$\vec{v}(t) = (v_x(t), v_y(t)) = (\frac{dx}{dt}, \frac{dy}{dt}) \tag{8}$$

$$\vec{a}(t) = (a_x(t), a_y(t)) = (\frac{dv_x}{dt}, \frac{dv_y}{dt}) = (\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}) \tag{9}$$

"Inverse relationships" among $\vec{r}$(t), $\vec{v}$(t), and $\vec{a}$(t): 2D motion:
You just have to apply the "inverse relationships" for 1-dimension (equations (3) and (4)) to each component here.

Example 2: Change in the position vector over time:
A bird is flying in the sky. It moves in 3-dimensions (i.e., it can go up/down, north/south, west/east). Suppose it's velocity is described by

$$\vec{U}(t) = (V_x\frac{T-t}{T}, V_y\frac{1}{\sqrt{1+kt}}, 0) \tag{10}$$

where V_x, V_y, T, and k are all constants (i.e. they don't change over time). Suppose that at $t = 0$, the bird is at position $(x, y, z) = (x_0, y_0, z_0)$ (here $x_0, y_0, and z_0$ are constants as well).

(**a**) What are the dimensions of all the constant parameters? [i.e. Do they have dimensions of length, or time, or mass?]
[Hint: To solve this problem, note that you can *only* add two quantities if they have the same dimensions. That is, you cannot add seconds to kg. You cannot add meters to seconds, etc. By this reasoning, you cannot add a variable (or a constant parameter) with a dimension of time (e.g. seconds) to a pure number. As an example, you cannot add the pure number "1" to 3 seconds.]

(**b**) What is the bird's position $\vec{r}(t) = (x(t), y(t), z(t))$ at time t?

(**c**) At what time t_f does the y-component of the bird's position become $2y_0$?

(**d**) Is the total distance travelled by the bird between time $t = 0$ and $t = t_f$ equal to $\sqrt{(x(t_f) - x_0)^2 + (y(t_f) - y_0)^2}$? Why or why not? [Hint: the answer is "no"! Make sure you understand why it isn't].

(**e**) Suppose now that the bird's velocity is $\vec{U}(t) = (V_x, 0, V_x\sqrt{t/T})$ and that $V_x > 0$. What is the total distance that the bird has travelled between time $t = 0$ and $t = T$?

Solutions:

(**a**) We have $(T - t)$ and we can only add or subtract two quantities with the same dimension. So T has dimension of time (because t does too). Then note that $(T - t)/T$ is *dimensionless* (i.e. it doesn't have any dimensions. It's a pure number). And since we need $V_x\frac{T-t}{T}$ to have dimension of length / time (because this is the x-component of the velocity), we must have V_x having a dimension of length / time. As for the y-component velocity, note that kt is being added to a pure number (the number "1"). Since a pure number is dimensionless, we must have kt being dimensionless too in order for us to add it to the "1". This means that k must have a dimension of 1/time. Finally, since $V_y\frac{1}{\sqrt{1+kt}}$ must have a dimension of length / time and $\frac{1}{\sqrt{1+kt}}$ is dimensionless pure number, V_y must have a dimension of length / time.

(**b**) We can solve component-by-component.

$$\begin{aligned}
\frac{dx}{dt} &= V_x\frac{T-t}{T} \\
\implies dx &= V_x\frac{T-t}{T}dt \\
\implies \int_{x_0}^{x} dx &= \int_0^t V_x\frac{T-t}{T}dt \\
\implies x - x_0 &= -\frac{V_x}{2T}(T-t)^2\Big|_0^t \\
\implies x(t) &= x_0 - \frac{V_x}{2T}[(T-t)^2 - T^2]
\end{aligned} \tag{11a}$$

$$\begin{aligned}
\frac{dy}{dt} &= V_y\frac{1}{\sqrt{1+kt}} \\
\implies dy &= V_y\frac{1}{\sqrt{1+kt}}dt \\
\implies \int_{y_0}^{y} dy &= \int_0^t V_y\frac{1}{\sqrt{1+kt}}dt \\
\implies y - y_0 &= \frac{2V_y}{k}\sqrt{1+kt}\Big|_0^t \\
\implies y(t) &= y_0 + \frac{2V_y}{k}[\sqrt{1+kt} - 1]
\end{aligned} \tag{12a}$$

And since the bird has zero velocity in the z-component, it will stay at z_0 at all times: $z(t) = z_0$. (you can also get this answer by calculating like above). Putting everything together, we have

$$\vec{r}(t) = (x_0 - \frac{V_x}{2T}[(T-t)^2 - T^2], y_0 + \frac{2V_y}{k}[\sqrt{1+kt} - 1], z_0) \tag{13}$$

(**c**) At t_f, we have $y(t_f) = 2y_0$. So,

$$
\begin{aligned}
y(t_f) &= 2y_0 \\
\implies y_0 &= \frac{2V_y}{k}[\sqrt{1+kt_f} - 1] \\
\implies t_f &= \frac{\left\{\frac{ky_0}{2V_y} + 1\right\}^2 - 1}{k}
\end{aligned}
\tag{14a}
$$

(**d**) The answer is no, the total distance travelled by the bird between $t = 0$ and $t = t_f$ is *not* $\sqrt{(x(t_f) - x_0)^2 + (y(t_f) - y_0)^2}$. The reason is that the bird's trajectory over time is a not a straight line. It is actually curved line. So we need to take this curvature into account. To see that the bird's path over time is not a straight line, note that the velocity vector $\vec{U}(t)$ is changing direction over time. To see this, write down the ratio of the y-component velocity $U_y(t)$ to x-component velocity $U_x(t)$:

$$
\frac{U_y(t)}{U_x(t)} = \frac{V_y\sqrt{1+kt}T}{V_x(T-t)}
\tag{15}
$$

Note that this ratio changes over time. That means the "slope" of the vector $(U_x(t), U_y(t))$ also changes over time. Thus the bird traces out a curved path. The correct answer for the total distance d_{total} is

$$
d_{total} = \int_0^{t_f} |\vec{U}(t)| dt
\tag{16}
$$

where $|\vec{U}(t)|$ is the speed of the bird at time t (it's the length of the velocity vector $\vec{U}(t)$ at time t).

(**e**) The bird travels for time T. We want to know the total distance it travels given that it's velocity as a function of time is now $\vec{U}(t) = (V_x, 0, V_x\sqrt{t/T})$. Again, by the same argument as above, we conclude that the bird traces out a curved path. If we know at what speed the bird travels along this curved path, then we can calculate the total distance. We first get the *speed* as a function of time t:

$$
\begin{aligned}
|\vec{U}(t)| &= \sqrt{V_x^2 + V_x^2\frac{t}{T}} \\
&= V_x\sqrt{1 + \frac{t}{T}}
\end{aligned}
\tag{17a}
$$

This speed changes as a function of time. The bird travels between time $t = 0$ and $t = T$. Thus the total distance d_{total} travelled during this time (i.e. the total lenth of the curved path) is

$$\begin{aligned} d_{total} &= \int_0^T |\vec{U}(t)| dt \\ &= \int_0^T V_x \sqrt{1 + \frac{t}{T}} dt \\ &= V_x \frac{2}{3} T (1 + \frac{t}{T})^{3/2} \Big|_0^T \\ &= \frac{2V_x T}{3} (\sqrt{8} - 1) \end{aligned} \tag{18a}$$

Circular motion: A **uniform circular motion** means that an object is going around in a perfect circle at a *constant speed*. It's important to note that this particle does *not* have a constant *velocity* because velocity is a vector; both its direction and length must remain unchanged for us to say that the particle's velocity remains constant. The particle is moving around at a constant speed means that the length of the velocity vector remains the same over time (i.e., speed = length of the velocity vector). But since the particle's moving in a circle, it's direction of motion is constantly changing. So the velocity is constantly changing. More specifically, we can see that the velocity vector rotates in circle itself. So the object accelerates (because the velocity vector changes over time). This is a special form of acceleration that we call **centripetal acceleration**. This is special because this acceleration only causes the direction of the object's velocity vector to constantly change over time but not the magnitude of the velcoity vector (i.e., its speed). The magnitude of the acceleration vector doesn't change (but it's direction does, in order to make the velocity vector rotate around in full circle, the centripetal acceleration vector must rotate around a full circle as well). Your textbook shows you how to calculate the magnitude of the centripetal acceleration vector (Pgs. 61- 62, 3rd edition). Let me show you a different way of deriving the same formula here.

Example 3: Calculating the centripetal acceleration for a uniform circular motion : A 2nd method - Different from the book: We will actually use the method used in Example 3. Here, the idea is that we can write the position of the object as a vector: $\vec{r}$(t) = (x(t), y(t)). Then by taking its derivative twice, we would get the acceleration vector $\vec{a}$(t). Then we're done! So the only difficult step here is determining what x(t) and y(t) are. To do this, let's draw a picture (almost always, draw a picture to help orient yourself in physics problems). Consider an object moving around at a constant speed in a circle of radius R with a constant speed v. Let's also say that initially ($t = 0$), the object is at the 12 o'clock position (considering the circle as a clock). Moreover, let's say that the object moves around clockwise, like an actual clock. We know that the object will sweep the same angle per unit time because it's moving at a constant speed around the circle. Let's say T is the **period**: the time the object takes to go around one full circle. Then we have

$$T = \frac{2\pi R}{v} \tag{19}$$

because $2\pi R$ is the distance that the object have covered after going around the circle once in time T. Moreover the angular speed (= angle swept out per unit time) in radians, which we call ω, is

$$\omega = \frac{2\pi}{T} \tag{20}$$

This has units of radians/time. Note that we always measure angles in radians instead of degrees in physics. This is just a convention, not for deep scientific reasons. If you really wanted to, then in degrees / time we would have $\omega = \frac{360°}{T}$, in degrees/time. But we'll keep the ω in radians/time. Note that ωt is in fact the angle that the object sweeps out in total in time t. To see this, note that when $t = T/2$, then the object must be at the 6 o'clock position. That is, it must have swept out π radians (i.e., 180°). That's exactly what ωt gives you. We have $\omega t = \omega\frac{T}{2} = \frac{2\pi}{T}\frac{T}{2} = \pi$. When $t = T/4$, we expect the particle to be at the 3 o'clock position (i.e., $\pi/2$ radians, which is 90°). Indeed, ωt also gives us $\pi/2$. So everything checks out. The position of the object $r(t)$ is thus

$$\vec{r}(t) = (x(t), y(t)) = (-Rsin(\omega t), Rcos(\omega t)) \tag{21}$$

The velocity $\vec{v}(t)$ is, by definition, the derivative of $r(t)$ with respect to time:

$$\begin{aligned}
\vec{v}(t) &= \frac{d\vec{r}}{dt} \\
&= (\frac{dx}{dt}, \frac{dy}{dt}) \\
&= (-R\omega cos(\omega t), -R\omega sin(\omega t)) \\
&= (v_x(t), v_y(t))
\end{aligned} \tag{22a}$$

where $v_x(t)$ is the x-component velocity vector and $v_y(t)$ is the y-component velocity vector. Finally, the acceleration is, again by definition, the following:

$$\begin{aligned}
\vec{a}(t) &= \frac{d\vec{v}}{dt} \\
&= (\frac{dv_x}{dt}, \frac{dv_y}{dt}) \\
&= (\omega^2 Rsin(\omega t), -\omega^2 Rcos(\omega t)) \\
&= (a_x(t), a_y(t))
\end{aligned} \tag{23a}$$

where a_x is the x-component acceleration vector and a_y is the y-component acceleration vector. First, let's calculate the length of the acceleration vector as a function of time. The length $|\vec{a}(t)|$ is

$$
\begin{aligned}
|\vec{a}(t)| &= \sqrt{a_x^2 + a_y^2} \\
&= \sqrt{\omega^4 R^2 (sin^2(\omega t) + cos^2(\omega t))} \\
&= \omega^2 R \sqrt{1} \\
&= \omega^2 R \\
&= \left\{\frac{2\pi}{T}\right\}^2 R \\
&= \left\{\frac{2\pi v}{2\pi R}\right\}^2 R \\
&= \frac{v^2}{R}
\end{aligned}
\tag{24a}
$$

So the length of the acceleration vector is constant (it doesn't change over time). This length is always $\frac{v^2}{R}$. We call this the **centripetal acceleration** (note that your book gives the same formula but derived in a different method). The length of the acceleration vector is constant over time. But the *vector* itself constantly changes over time. We can visualize how the acceleration vector changes over time by computing the *dot product* between $\vec{v}$ and $\vec{a}$:

$$
\begin{aligned}
\vec{v}(t) \cdot \vec{a}(t) &= (v_x(t), v_y(t)) \cdot (a_x(t), a_y(t)) \\
&= v_x(t)a_x(t) + v_y(t)a_y(t) \\
&= -R^2\omega^3 cos(\omega t) sin(\omega t) + R^2\omega^3 sin(\omega t) cos(\omega t) \\
&= 0
\end{aligned}
\tag{25a}
$$

We also know, by definition of dot product between two vectors, that

$$
\vec{v}(t) \cdot \vec{a}(t) = |\vec{v}(t)||\vec{a}(t)| cos(\phi) \tag{26}
$$

where ϕ is the angle between the two vectors $\vec{v}(t)$ and $\vec{a}(t)$. Since $\vec{v}(t) \cdot \vec{a}(t) = 0$, this means that $cos(\phi)$ =0 (this must be the case since the two vectors have non-zero lengths). This in turn means that $\phi = \pi/2$ (in radians). This is 90° in degrees. We know that the velocity vector is always tangential to the circle. So $\vec{a}$ must either always point towards the center or always away from the center. By computing the dot product between $\vec{r}$ and $\vec{a}$, we can see that the $\vec{a}$ must be pointing towards the center of the circle (not away from it) at all times (check this for yourself – You will need to know why getting $\vec{r}(t) \cdot \vec{a}(t) =$ -1 and the fact that $\vec{r}(t)$ is a vector that points out of the center of the circle at (0, 0) imply that $\vec{r}(t)$ and $\vec{a}(t)$ are parallel to each other and point in opposite directions).

Example 4: Two dimensional motion with variable velocity:

Caenorhabditis elegans is a $\sim$ 1 mm long roundworm that is a model organism. We call it *C. elegans* for short. It is a useful organism to study animal development and for neurosceince because it is completely transparent (so you can look at every cell inside the worm under a microscope) and consists of exactly 1031 cells (at least the males) which is a small enough number that researchers can locate and characterize the function each of the cells in the worm. Some forms of this worm (called the *hermaphrodite*) because it consists of exactly 302 neurons and the connections among these neurons are all known. In summary, it's a remarkable organism. The worm also executes stereotyped movements. The worm crawls on a surface like a sinusoidal wave. Suppose we look at the head of the worm. It will move with velocity

$$\begin{aligned} \vec{v}(t) &= (v_0, v_0 cos(\omega t)) \\ &= (v_x(t), v_y(t)) \end{aligned} \tag{27a}$$

This velocity vector has two components: x-component velocity v_x (horizontal component of motion) and y-component velocity v_y (vertical component of motion). Note that the horizontal component of velocity is constant ($v_x(t) = v_0$). The vertical component of velocity varies over time as a cosine function ($v_y(t) = v_0 cos(\omega t)$). Let's say the head of the worm is at the location (x, y) = (0, 0) at $t = 0$.

(**a**) What is the position of the worm's head at time t?

(**b**) What is the *distance* travelled by the worm after time t? You'll get a difficult integral here. You can leave your final answer as an unsolved integral). Also, what is the *displacement* of the worm's head after time t relative to its starting position (at $t = 0$)? [Note the difference between *distance* and *displacement*.]

(**c**) Watching the worm's motion over a long time, what would you say is the *average velocity* of the worm's head?

Solutions:

(**a**) The position $\vec{r}(t) = (x(t), y(t))$ at time t is a vector with two components. Like any two dimensional vector, we can break it up into the x-component ($x(t)$) and the y-component ($y(t)$). $x(t)$ and $y(t)$ do not "interact" or "interfere" with each other because they are orthogonal (independent) of each other. Practically, this means that we can solve for $x(t)$, and $y(t)$ separately, and then put the two results together to get $r(t)$.

$$\begin{aligned} x(t) &= v_x(t) \cdot t \\ &= v_0 t \end{aligned} \tag{28a}$$

So we have $x(t)$. Now, let's find $y(t)$. This one is more tricky because $v_y(t)$ varies over time. Keeping in mind that by definition, $dy/dt = v_y(t)$, we have

$$\begin{aligned}
&\frac{dy}{dt} = v_y(t) \\
&\implies dy = v_y(t)dt \\
&\implies \int_0^y dy = \int_0^t v_0 cos(\omega t)dt \\
&\implies y(t) - 0 = v_0 \frac{sin(\omega t)}{\omega} \\
&\implies y(t) = \frac{v_0}{\omega} sin(\omega t) \qquad (29a)
\end{aligned}$$

So putting $x(t)$ and $y(t)$ together, we have

$$\vec{r}(t) = (v_0 t, \frac{v_0}{\omega} sin(\omega t)) \qquad (30)$$

as the position of the worm's head at time t.

(**b**) First, the *displacement* of the worm's head at time t relative to its starting position (i.e. position at $t = 0$) is just $\vec{r}(t)$ [Remember: "displacement" at time t relative to the initial position means the location at t relative to initial location]. The more tricky part is calculating the total *distance* travelled by the worm's head. First, note that the distance depends on the *speed* (which is a number, not a vector). The speed is the length of the velocity vector: $|\vec{v}|$:

$$\begin{aligned}
|\vec{v}(t)| &= \sqrt{v_0^2 + v_0^2 cos^2(\omega t)} \\
&= v_0 \sqrt{1 + cos^2(\omega t)} \qquad (31a)
\end{aligned}$$

Thus the *speed* at time t is $v_0\sqrt{1 + cos^2(\omega t)}$. It changes as a function of time. The total distance d_{total} travelled after time t is

$$\begin{aligned}
d_{total} &= \int_0^t |\vec{v}(t)| dt \\
&= v_0 \int_0^t \sqrt{1 + cos^2(\omega t)} dt \qquad (32a)
\end{aligned}$$

This is a difficult integral. We leave it as it is.

(**c**) Suppose we watch the worm moving for a long time T. We know that the worm head's velocity is constantly changing over time. But when you watch the worm for a long time, you "feel" that it's moving at some "constant" velocity (that is, you can ignore how the worm's velocity quickly varies every second if you're interested in the gross movement of the worm over a time period of several hours). This constant velocity is what we call an *average velocity* $\vec{v}_{avg}$. It has the property that $\vec{v}_{avg}$T must equal the total displacement after time T. $\vec{v}_{avg}$ is a vector so we can break it up into x-component and y-components: $\vec{v}_{avg} = (v_{x,avg}, v_{y,avg})$. To get $v_{x,avg}$, we note that

$$\begin{aligned} v_{x,avg}T &= v_0 T \\ \implies v_{x,avg} &= v_0 \end{aligned} \tag{33a}$$

No surprise here. The x-component velocity is constant over time so the x-component of the average velocity is also the same constant v_0. The y-component of the average velocity is more tricky. We have

$$\begin{aligned} v_{y,avg}T &= \int_0^T v_0 cos(\omega t)dt \\ \implies v_{y,avg} &= \frac{1}{T}\int_0^T v_0 cos(\omega t)dt \end{aligned} \tag{34a}$$

There is a natural value to pick for T, which is just the time it takes for the cosine function to go around once (i.e. $\omega T = 2\pi$; in other words, pick T to be the **period**). In this case, we get

$$\begin{aligned} v_{y,avg} &= \frac{1}{T}\int_0^T v_0 cos(\omega t)dt \\ &= \frac{v_0}{T\omega}(sin(\omega T) - sin(0)) \\ &= \frac{v_0}{T\omega}(0-0) \\ &= 0 \end{aligned} \tag{35a}$$

$v_{y,avg} = 0$. This makes sense because within one cycle of the cosine, the y-component velocity v_y has positive values that equally cancel out the negative values (due to the nature of cosine function). By the same reasoning, we would get the same answer even if we picked T to be $2\times$ period, or $3\times$ period, or any integer times the period. That is, if $T = n\tau$, where τ is one period of the cosine function and n is an integer (i.e. $\tau\omega = 2\pi$), we would get

$$\begin{aligned} v_{y,avg} &= \frac{1}{T}\int_0^T v_0 cos(\omega t)dt \\ &= \frac{v_0}{n\tau\omega}(sin(n\tau\omega) - sin(0)) \\ &= \frac{v_0}{T\omega}(0-0) \\ &= 0 \end{aligned} \tag{36a}$$

Thus the same answer when $T = n\tau$. Combining $v_{x,avg}$ and $v_{y,avg}$, we have

$$\vec{v}_{avg} = (v_0, 0) \tag{37}$$

This makes sense because after a long time T, the worm's head has moved equal amount upwards in y as it does downwards in y. Thus the y-component of displacement is zero. The worm is always moving forward in the +x direction, on average, at speed v_0.

Example 5: Chemotaxis of the bacterium, *E. coli*:
The bacterium *Escherichia coli (E. coli)* finds its food (e.g., sugars, nutrients) by "smelling" its way towards food. An analogy is that you smell a perfume because your friend sprayed a lot of it on himself /herself. With your eyes closed, you walk around the room while constantly sniffing. If the scent becomes stronger, then you know that you are walking in the right direction. This way, you can eventually find your friend with your eyes closed. . If the smell gets weaker, you know that you are moving away from your friend. In that case, you would change the direction in which you walk. You randomly pick in which direction you should walk towards. You check each time if the smell is getting stronger or weaker. If it's getting weaker, you stop walking and then again randomly pick a new direction and then walk in that direction. You can repeat this until you pick a direction in which the smell gets stronger. If the smell is getting stronger, you keep walking in that direction without changing your direction of walk. Amazingly, the single-celled microscopic bacterium, *E. coli*, does the same thing. It doesn't have a nose like us but it has receptors on its cell membrane that can bind "smell" molecules (like the "smell" receptors inside our nose bind the perfume molecules and communicate it to our brain – that's in fact how we identify the smell). In *E. coli*, these receptors are called "chemoreceptors". A *chemo-attractant* is a molecule that the *E. coli* likes and is thus drawn towards (such as food: e.g., aspartamine). The chemoreceptors that are bound to a chemo-attractant sends a signal to the cell's nucleus and other inner parts, to coordinate the cell's movement (just like the activated smell receptors in your nose send a signal to your brain). As the *E. coli* senses higher concentration of the chemoattractant, it keeps swimming in that direction. If it senses that the concentration of the chemoattractant is decreasing, it stops swimming in that direction, then randomly chooses a new direction, and then swims in this new direction.

Let's consider a simple model of chemotaxis. Suppose our *E. coli* can only swim along a line. That is, it can either move to the right or move to the left.

(**a**) In the absence of food, the *E. coli* does not prefer one direction over the other direction. It swims to the left at a constant speed V during a time interval ΔT, and then it immediately turns to the right, and then swims at the same constant speed V during time interval ΔT. The *E. coli* repeats this protocol over and over again. After watching the cell moving for a long time, what would you say is the average velocity of the *E. coli*?

(**b**) Now there's food on the right side. Smelling the diffusing chemoattract molecules coming from its right, the *E. coli* now swims to the right for a longer period of time than to the left. Let's say that the *E. coli* is initially at $x = 0$. The food is at $x = 4V\Delta T$. Suppose the *E. coli* initially swims to the right for time interval $2\Delta T$ at a constant speed V. Afterwards, it immediately turns to the left, and then swims to the left at a constant speed V for time interval ΔT. Afterwards, it immediately turns to the right, and swims to the right at constant speed V for time duration $2\Delta T$ again. It then turns to the left, and then swims at constant speed V for time duration ΔT. And then it repeats this back-and-forth, over and over again until it reaches the food. When does the *E. coli* reach the food? Assume that the *E. coli* first starts moving to the right at $t = 0$.

(**c**) Same scenario as in (b) but now say the food is very far away from the origin (let's just say that the food is at $x = \infty$). Say you watch for a long time the *E. coli* moving as in (b). What would you say is the average velocity of the *E. coli*? Also, after a long time, where is the *E. coli*?

(**d**) Now, a different scenario. Suppose the *E. coli* swims straight from $x = 0$ to some food at $x = L$, without ever turning back. Moreover, as the *E. coli* swims closer to the food, the concentration of the chemoattractant increases, meaning that the *E. coli*'s receptors bind more of the chemoattractants. Let's say that this causes the *E. coli* to swim faster (so it accelerates over time as it gets closer to the food). Concretely, say that its velocity is $v(t) = v_0 e^{t/T}$, where T is some constant time and $v_0 > 0$. What is the average velocity of the *E. coli* between time $t = 0$ and $t = T$?

Solutions:

(**a**) The average velocity is a vector. In one dimension, this means that it can be either a positive number or a negative number (negative number means that the average velocity vector points to the left, and a positive number means that the average velocity vector points to the right). Intuitively, you might be able to guess that the average velocity must be zero, because the E. coli moves to the right at the same speed for an equal amount of time as it moves to the left at the same speed. Mathematically, we can calculate this as follows. Suppose we watch the E. coli for a time $t_{total} = N\Delta T$ (to make it easy for ourselves, let's say N is an even number). Then we know that the cell must go to the left $N/2$ times and to the right $N/2$ times. So the average velocity is

$$\begin{aligned} V_{avg} &= \frac{V(N/2)\Delta T + (-V)(N/2)\Delta T}{N\Delta T} \\ &= 0 \end{aligned} \tag{38a}$$

Above is just a formula for computing an average. Note that we have (-V) to indicate *velocity* vector that points to the left (- direction), and (+V) to indicate *velocity* vector to the right (+ direction). So indeed, the average velocity is zero.

(**b**) We can just write out the sequence of events:

$$4V\Delta T = 2V\Delta T - V\Delta T + 2V\Delta T - V\Delta T + 2V\Delta T \tag{39}$$

So the total time Δt_{total} taken to reach the food at $x = 4V\Delta T$ is

$$\begin{aligned} \Delta t_{total} &= 2\Delta T + \Delta T + 2\Delta T + \Delta T + 2\Delta T \\ &= 8\Delta T \end{aligned} \tag{40a}$$

(**c**) As in (a), we average. For convenient, we pick the "long time" of observation (call it

τ) to be $\tau = N\Delta T$ (for some large even integer N). Then as in (a), we average:

$$\begin{aligned} V_{avg} &= \frac{V(N/2)(2\Delta T) + (-V)(N/2)\Delta T}{N\Delta T} \\ &= \frac{2V - V}{2} \\ &= \frac{V}{2} \end{aligned} \tag{41a}$$

So the average velocity V_{avg} is V/2. Note that this is a positive number because V (since it's a speed) is a positive number (Speed *cannot* be a negative number). This mean that on average, the *E. coli* moves to the right at an *average speed* of $V/2$, which is smaller than V. This makes sense because the progress that the *E. coli* makes in moving to the right is counteracted by its motion to the left. After a long time t, we can say that the *E. coli* is at a position approximately equal to

$$x(t) = V_{avg}t = \frac{Vt}{2} \tag{42}$$

(**d**) The velocity is $v(t) = v_0 e^{t/T}$. Between $t = 0$ and $t = T$, the average velocity is

$$\begin{aligned} v_{avg} &= \frac{1}{T}\int_0^T v_0 e^{t/T} dt \\ &= \frac{v_0}{T} T e^{t/T}\Big|_0^T \\ &= v_0(e - 1) \end{aligned} \tag{43a}$$

Above is just the formula for computing an average. One way to see this is by reminding yourself what an integral is. Note that we can think of a timeline, going from $t = 0$ to $t = T$, then chopping it into very tiny (infinitesimal) time intervals. Say we chop it into N segments (N is a very large number). Let each time interval be of length Δt. Then we have $\Delta t = T/N$. The idea is that we'll make N so large that Δt is very small, so we can write $\Delta t = dt$. Then our average formula in part (c) becomes

$$\begin{aligned} v_{avg} &= \frac{v_0 e^{t_1/T} dt + v_0 e^{t_2/T} dt + ... + v_0 e^{t_N/T} dt}{T} \\ &= \frac{1}{T}\int_0^T v_0 e^{t/T} dt \end{aligned} \tag{44a}$$

Above is just the definition of the integral.