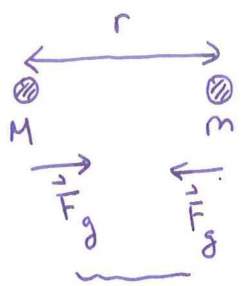
Lecture 5

Newton's law of gravity.



still follows Newton's 3rd law (of course)

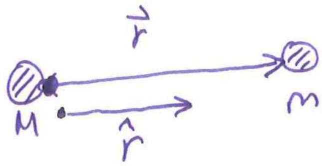
$$F_g = |\vec{F}_g| = \frac{GMm}{r^2}$$

$G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$

"Constant of universal gravitation"

- Gravity is an attractive force
- Two objects attract each other along a straight line that joins them.

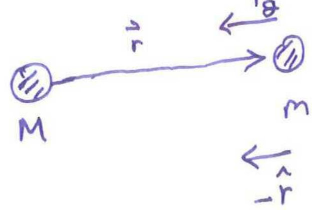
Direction of force :



$\vec{r} = r\,\hat{r}$ $\quad (r = |\vec{r}|)$

$|\hat{r}| = 1$ $\quad \hat{r}$ is a unit vector like $\hat{x}, \hat{y}, \hat{z}$, $\hat{i}, \hat{j}, \hat{k}$.

The force on m is :

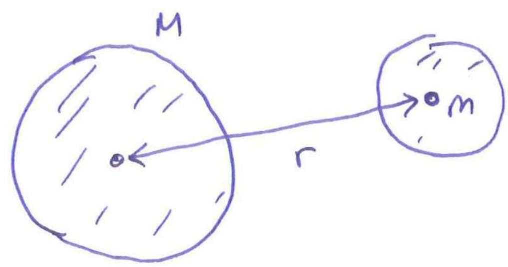


$$\vec{F}_g = -\frac{GMm}{r^2}\,\hat{r}$$

(in $-\hat{r}$ direction)

because attracted towards M.)

- Circular orbit :



Two planets of masses M and m. Assume that both planets are spheres with uniform mass density. (i.e. Density of the large planet is : $\rho = \frac{M}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$)

In such cases, gravitational force of one body on the other is still

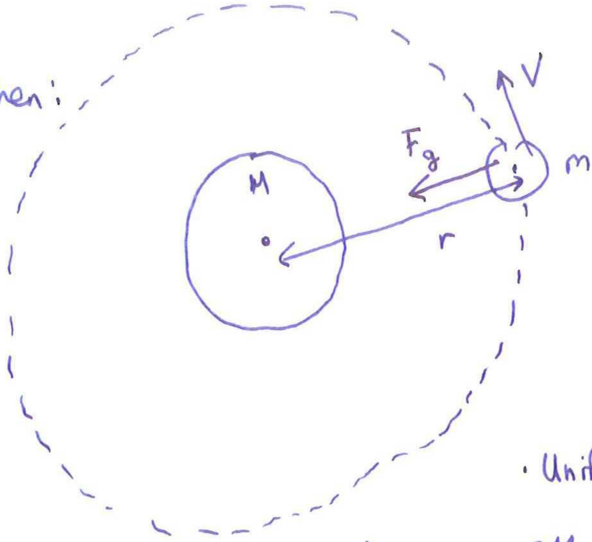
$$F_g = \frac{GMm}{r^2}$$

r = distance between centers of planets.

Where R = radius of the large planet.

Say the small planet is orbiting around the big planet. Assume that the big planet remains stationary. (Someone's holding the big planet by hand.)

Then:



- m orbits around M at constant speed V.
- circle of radius r.

$$F_g = \frac{GMm}{r^2}$$

- Uniform circular motion. so F_g = centripetal force. $= \frac{mv^2}{r}$

$$\Rightarrow \frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}v^2}{r}$$

$$\Rightarrow V = \sqrt{\frac{GM}{r}}$$

~~circ~~ speed at which m ~~orbits~~ moves in the circular orbit.

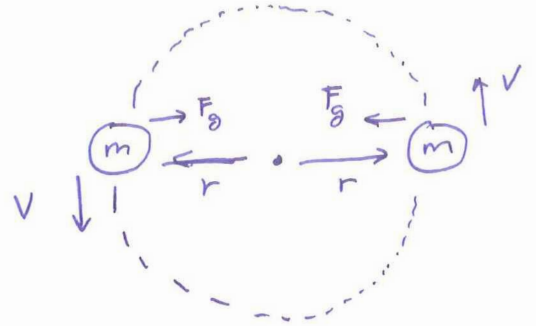
- Period of orbit T = time taken to complete ~~the~~ one circle.

$$VT = \underbrace{2\pi r}_{\text{circumference of the circle.}}$$

$$\Rightarrow T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}}$$

$$= \sqrt{\frac{4\pi^2 r^3}{GM}}$$

EX: Now suppose 2 identical planets orbit around each other.



$$F_g = \frac{Gm^2}{(2r)^2} \quad \leftarrow \text{gravitational attraction between them.}$$

$$= \frac{Gm^2}{4r^2}$$

Centripetal motion $\Rightarrow$ centripetal force: $\dfrac{mv^2}{r}$ ← r radius of orbit.

$$\Rightarrow \quad \frac{mv^2}{r} = F_g = \frac{m^2 G}{4r^2} \quad \Rightarrow \quad v^2 = \frac{Gm}{4r} \quad \Rightarrow \quad v = \frac{1}{2}\sqrt{\frac{Gm}{r}}$$

↑ speed of orbit.

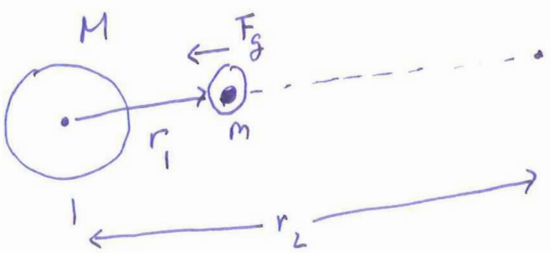
so
$$T = \frac{2\pi r}{v} = 2\pi r \cdot 2\sqrt{\frac{r}{Gm}}$$

$$= \boxed{4\pi\sqrt{\frac{r^3}{Gm}}} \quad \leftarrow \text{period of orbit.}$$

Gravitational energy:

- How much work do you do on an object of mass m to push it out from r_1 to r_2?

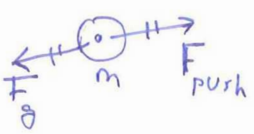
$$F_g = \frac{GMm}{r^2}$$



- To push out the mass to r_2 away from M, you have to do work against gravity

$\Rightarrow$ (you do positive work)

$$W_{1\to 2} = \int_{r_1}^{r_2} \frac{GMm}{r^2}\, dr$$



$F_{push} = F_g$.

$$= \frac{-GMm}{r}\Bigg|_{r_1}^{r_2} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

So: $W_{1\to 2} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

Note that when $r_2 > r_1$: $\frac{1}{r_2} - \frac{1}{r_1} < 0$.

$\Rightarrow W_{1\to 2} > 0$. (Sanity check! yes, this makes sense because you have to overcome gravity so you do positive work.)

- To push the mass m to very far away:

(i.e. $r_2 \to \infty$), starting from r_1 away from M, we have

$$W_{1\to 2} \underset{r_2\to\infty}{=} -GMm\left(\underbrace{\frac{1}{r_2}}_{0} - \frac{1}{r_1}\right) = \frac{GMm}{r_1}$$

← work that you do on the mass to bring the mass out to ∞.

- To define the gravitational potential energy:

Note from above that $W_{1\to 2} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

$$= \frac{-GMm}{r_2} - \left(-\frac{GMm}{r_1}\right)$$

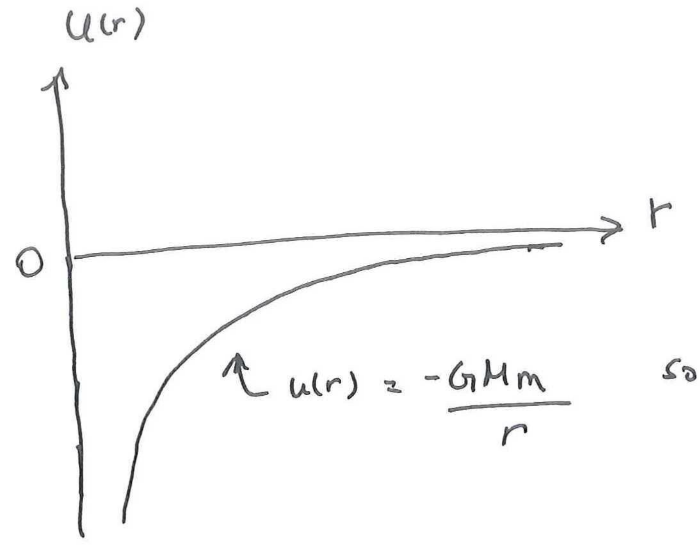
we know from work-Energy theorem that

$\Delta U_{1\to 2} = W_{1\to 2}$ ← change in potential energy in going from 1 to 2 equals to the work that you put into the system.

so: $\Delta U_{1\to 2} = U_2 - U_1$

$$= \frac{-GMm}{r_2} - \left(\frac{-GMm}{r_1}\right)$$

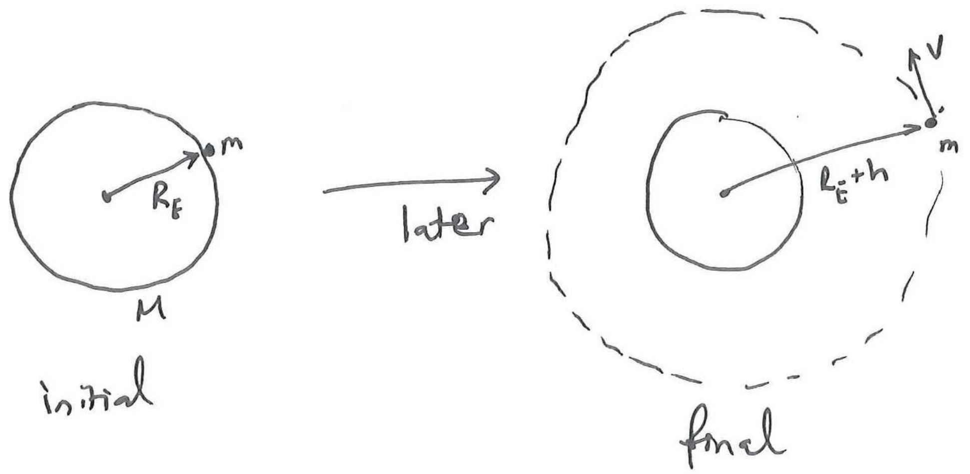
So we define $\boxed{U(r) = \frac{-GMm}{r}}$ ← gravitational potential energy



so as you go towards $r \to \infty$,
$U(r) \to 0$
(And $U(r)$ increases as $r \to \infty$
as it should.)

Ex A rocket is on surface of Earth (Earth radius = R_E) at rest.
The rocket is launched and put into circular orbit around the Earth at speed V, at height h above the surface of Earth.
How much work is done to put the rocket into the circular orbit? m = mass of rocket M = mass of Earth.

Sol'n:



Initial: $E_{total,initial} = PE_{initial} + \cancel{KE_{initial}}^{\,0}$

$$= -\frac{GMm}{R_E}$$

Final: $E_{total,final} = PE_{final} + KE_{final}$

$$= -\frac{GMm}{R_E+h} + \frac{mv^2}{2}$$

Now, centripetal motion $\Rightarrow \dfrac{\cancel{m}v^2}{R_E+h} = \dfrac{GM\cancel{m}}{(R_E+h)^2}$

$$\Rightarrow v^2 = \frac{GM}{R_E+h}$$

so: $\dfrac{mv^2}{2} = \dfrac{GMm}{2(R_E+h)}$ ← kinetic energy in orbit.

so: $E_{\substack{total\\final}} = \dfrac{-GMm}{2(R_E+h)}$ ← final total energy.

$\therefore$

$$\Delta E = E_{\substack{total\\final}} - E_{\substack{total\\initial}}$$

$$= \frac{-GMm}{2(R_E+h)} + \frac{GMm}{R_E}$$

$$= GMm\left[\frac{-1}{2(R_E+h)} + \frac{1}{R_E}\right]$$

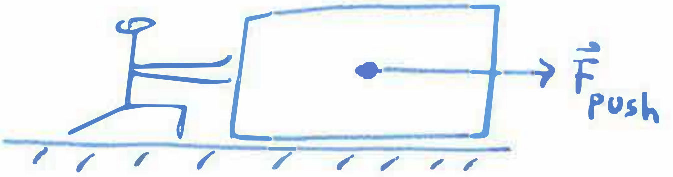
$$= GMm\left[\frac{-R_E + 2(R_E+h)}{2R_E(R_E+h)}\right]$$

$$= GMm\left[\frac{R_E+2h}{2R_E(R_E+h)}\right]$$

$$= \frac{GMm}{2R_E^{\cancel{2}}} \cdot \frac{\cancel{R_E}(1+2h/R_E)}{1+h/R_E}$$

$$= \boxed{\frac{GMm}{2R_E}\,\frac{[1+2h/R_E]}{1+h/R_E}}$$ ← Amount of work required to put rocket into orbit.

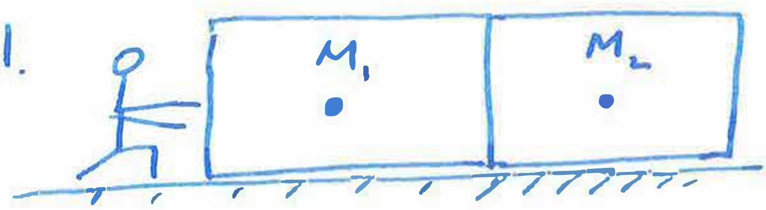
Center of mass



- Block being pushed by a person with force $\vec{F}_{push}$.
- $\vec{F}_{push}$ considered to act on a point that represents the block.
- But the block is made up of billions and billions of particles (atoms). Force must act on all of the atoms in the blocks for the entire block to move forward.
  So why can we think of the block as just a single object represented by a point?
- The answer lies in the concept of "center of mass".

To derive the concept of center of mass, let's do some examples.

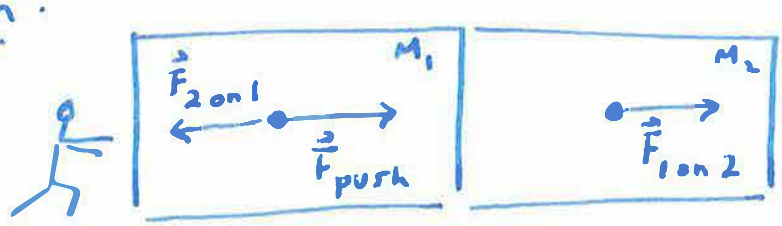
EX 1.



- No friction with ground
- 2 blocks of masses M_1 and M_2.
- You push on block 1 (M_1) with force $\vec{F}_{push}$.

- Question: How do the blocks accelerate?

- Solution:



- $\vec{F}_{1 on 2}$ = force that block 1 exerts on block 2.
- $\vec{F}_{2 on 1}$ = force that block 2 exerts on block 1 (reaction force ← Newton 3rd law)

- **Logic:** You push on M_1 with force $\vec{F}_{push}$
  - M_1 pushes on M_2 with force $\vec{F}_{1\,on\,2}$ to make M_2 move.
  - M_2 pushes back on M_1 with force $\vec{F}_{2\,on\,1}$

    (back reaction force ← Newton's 3rd law)

Note that by Newton's 3rd law: $\vec{F}_{1\,on\,2} = -\vec{F}_{2\,on\,1}$

(i.e. 2 forces are equal in magnitude, opposite in directions.)

So:

Block 1: $M_1 A_1 = F_{push} - F_{2\,on\,1}$

Block 2: $M_2 A_2 = F_{1\,on\,2}$

Here, we're using the sign convention:

$\ominus \longleftrightarrow \oplus$

left   right

[Note that in 1-dimension, we can ~~don~~ use $\oplus$ / $\ominus$ sign to denote the direction of vectors.]

Can we determine A_1 & A_2?

~~[illegible]~~

Let's add above 2 equations together. Then we get:

$$M_1 A_1 + M_2 A_2 = F_{push} \underbrace{- F_{2\,on\,1} + F_{1\,on\,2}}_{\text{cancel each other.}}$$

$$\therefore \quad M_1 A_1 + M_2 A_2 = F_{push}$$

We know that M_1 & M_2 move together.

So $A_1 = A_2$. Let's call this $A = A_1 = A_2$

So:
$$(M_1 + M_2) A = M_1 A_1 + M_2 A_2$$
$$= F_{push}$$

So:
$$\boxed{A_1 = A_2 = A = \frac{F_{push}}{M_1 + M_2}}$$
← Acceleration of both blocks.

Technically, we're done. We answered the question. But let's look at the problem differently.

We have:

$$(M_1 + M_2) A = M_1 A_1 + M_2 A_2$$

$$\Rightarrow \boxed{A = \frac{M_1 A_1 + M_2 A_2}{M_1 + M_2} \quad \ldots\ [1]}$$

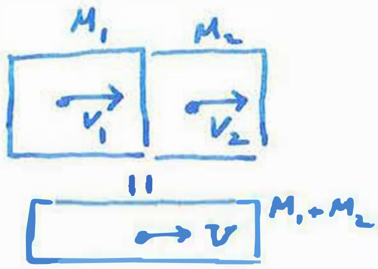
Acceleration of the "system" made of 2 blocks.

Note that $A_1 = \frac{dV_1}{dt}$, $A_2 = \frac{dV_2}{dt}$, $A = \frac{dV}{dt}$

V_1 = velocity of block 1 $\quad$ V_2 = velocity of block 2.

V = velocity of the system (i.e. system = 2 blocks joined together)

So we have, equation [1] becoming:



$$\frac{dV}{dt} = \frac{M_1 \frac{dV_1}{dt} + M_2 \frac{dV_2}{dt}}{M_1 + M_2}$$

$$\Rightarrow \boxed{\frac{dV}{dt} = \frac{d}{dt}\left[\frac{M_1 V_1 + M_2 V_2}{M_1 + M_2}\right] \quad \ldots\ [2]}$$

Solving for V, we get:

$$\boxed{V = \frac{M_1 V_1 + M_2 V_2}{M_1 + M_2} \quad \ldots\ [3]}$$

~~← Makes sense.~~

Exercise 1 →

Show how we get [3] from [2].

Hint: Use your knowledge on differential equations from Analysis course. Recall that if you have 2 functions: $f(t)$ and $g(t)$, then

$$\frac{df}{dt} = \frac{dg}{dt} \Rightarrow \frac{d}{dt}(f-g) = 0$$

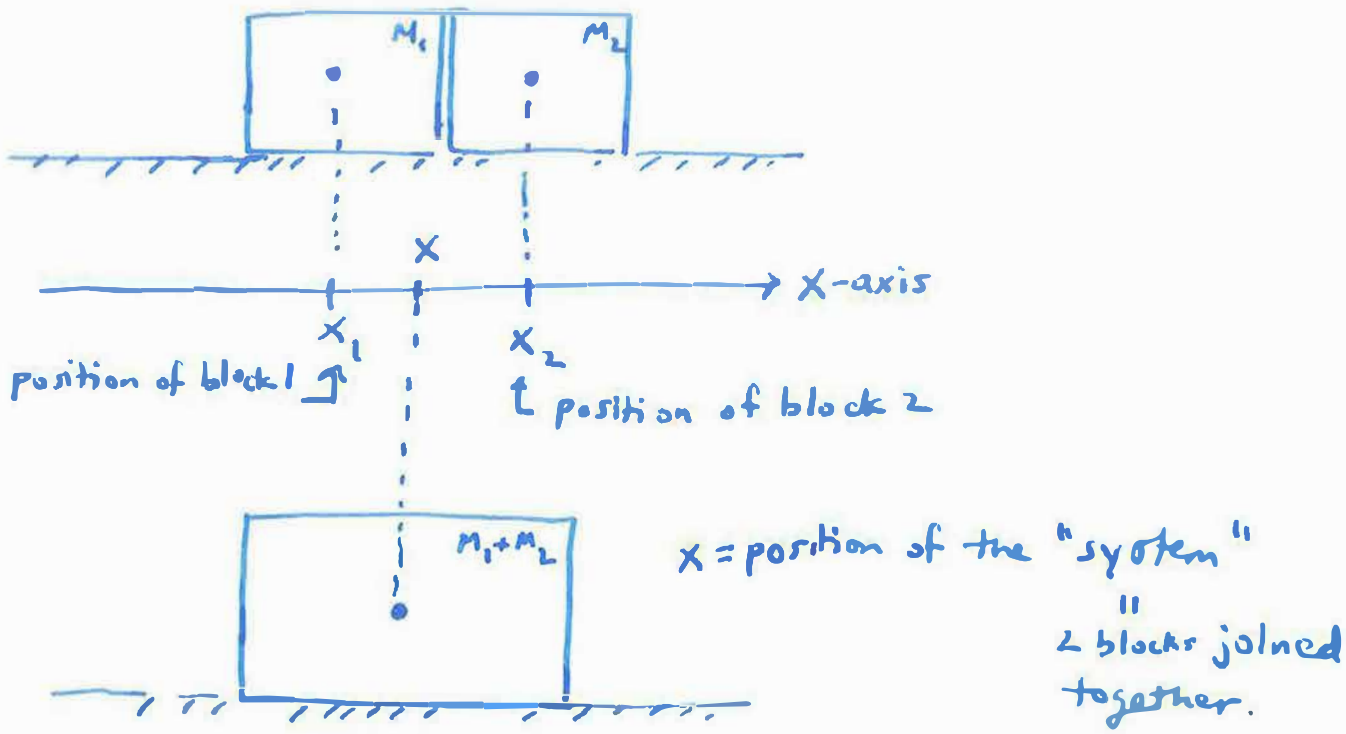
$$\Rightarrow f - g = c$$

c some constant.

Use this fact along with the fact that when $V_1 = 0$ and $V_2 = 0$, we must have $V = 0$.

Applying the same logic to equation [3], we have:

$$V = \frac{dX}{dt}, \quad V_1 = \frac{dx_1}{dt}, \quad V_2 = \frac{dx_2}{dt}$$



So eq'n [3] becomes:

$$\frac{dX}{dt} = \frac{M_1 \frac{dx_1}{dt} + M_2 \frac{dx_2}{dt}}{M_1 + M_2}$$

$$\Rightarrow \frac{dX}{dt} = \frac{d}{dt}\left[\frac{M_1 x_1 + M_2 x_2}{M_1 + M_2}\right]$$

Again, applying the same logic to get [3] from [2] on the previous page, ~~at above equation~~ we solve for X in above equation to get:

$$\boxed{X = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2} \quad \ldots \quad [4]}$$

position of the "system"

"

2 blocks joined together.