solutions to midterm exam 2016

Problem 1:

(a) Before collision: $\vec{P}_{\substack{total\\before}} = mV$

After: $\vec{P}_{\substack{total\\after}} = Mu$ $\qquad \vec{P}_{\substack{total\\after}} = \vec{P}_{\substack{total\\before}}$

$$\Rightarrow Mu = mV$$

$$\Rightarrow \boxed{u = \frac{m}{M}V} \quad \text{(to the right)}$$

velocity of M.

(b) Elastic collision $\Rightarrow \frac{mV^2}{2} = \frac{Mu^2}{2}$

$$\Rightarrow mV^2 = M\left(\frac{m}{M}V\right)^2$$

$$\Rightarrow m = \frac{m^2}{M}$$

$$\Rightarrow 1 = \frac{m}{M}$$

$$\Rightarrow \boxed{M = m}$$

(c) $\boxed{V_{CM} = \frac{mV}{m+M}}$ ← velocity of center of mass.

(d) The total momentum of the system, as measured by a person moving at the center of mass velocity is zero.

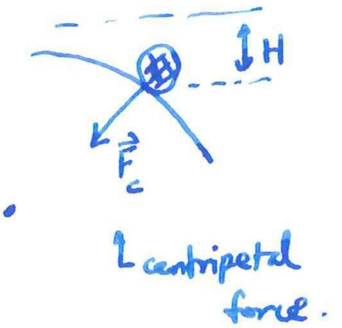
Because, $\vec{P}_{total} = (M+m)\,\vec{u}_{cm}$

CM velocity measured by person moving at CM velocity ($\vec{u}_{cm} = 0$)

so $\boxed{\vec{P}_{total} = 0}$ ← true before and after collision

Problem 2

(a)



$$|\vec{F}_c| = \frac{mv^2}{R}$$

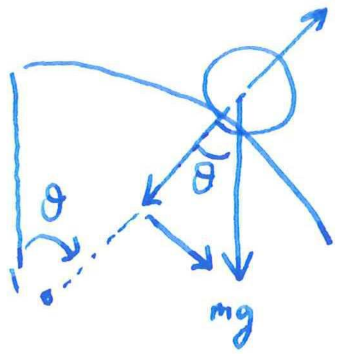
$$\frac{mv^2}{2} = mgH$$

$$\Rightarrow mV^2 = 2mgH$$

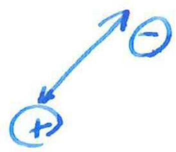
$$\Rightarrow \boxed{\frac{mV^2}{R} = \frac{2mgH}{R}}$$

F_c ← magnitude of centripetal force

(b)

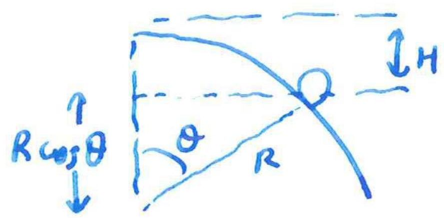


$$F_c = mg\cos\theta - F_N \qquad (F_N > 0)$$



$$\frac{2mgH}{R} = mg\cos\theta - F_N$$

$$\Rightarrow \quad F_N = mg\cos\theta - \frac{2mgH}{R} \quad \ldots (1)$$



$$H + R\cos\theta = R$$

so $\dfrac{R-H}{R} = \cos\theta$

so (1) becomes:

$$F_N = mg\left(\frac{R-H}{R}\right) - \frac{2mgH}{R}$$

$$\Rightarrow \boxed{F_N = mg - \frac{3mgH}{R}}$$

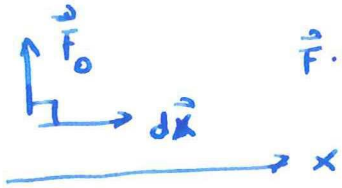
(c) Block leaves surface when $F_N = 0$ (because no longer in contact with surface)

so: $mg = \dfrac{3mgH}{R} \Rightarrow \boxed{H = R/3}$

Problem 3

(a) $\boxed{W_{total} = 0}$ because force must have some component in the direction of displacement to have non-zero work.

$$W_{total} = \int \underbrace{\vec{F}_0 \cdot d\vec{x}}_{0} = 0$$

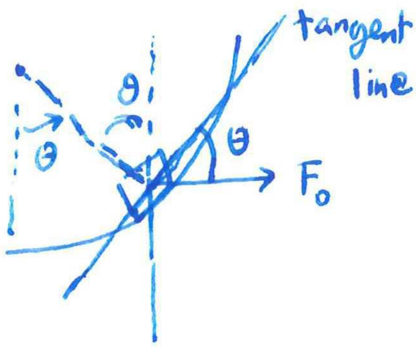


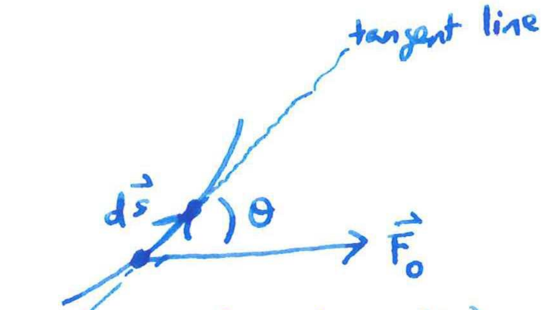
$$\vec{F} \cdot d\vec{x} = |\vec{F}|\, dx \underbrace{\cos(\pi/2)}_{0} = 0.$$

(b) $$W_{total} = \int \vec{F}_0 \cdot d\vec{x} = \int_0^L F_0\, dx = \boxed{F_0 L}$$

(c) $$W_{total} = \int_0^L F(x)\, dx = \left.\frac{a x^2}{2}\right|_0^L = \boxed{\frac{a L^2}{2}}$$

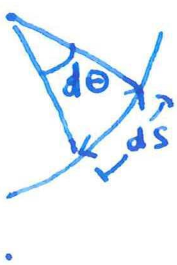
(d)



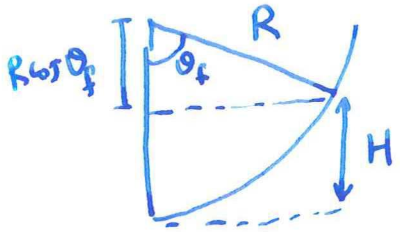


So $\vec{F}_0 \cdot d\vec{s} = F_0 (ds) \cos\theta$

so $$W_{total} = \int \vec{F} \cdot d\vec{s} = \int F_0 \cos\theta \, ds = \int_0^{\theta_f} F_0 \cos\theta \; R\, d\theta = F_0 R \sin\theta_f$$



$ds = R\, d\theta$



$$R\cos\theta_f + H = R.$$

$$\Rightarrow R^2\cos^2\theta_f = (R-H)^2$$

$$\Rightarrow R^2(1-\sin^2\theta_f) = R^2 - 2HR + H^2$$

$$\Rightarrow 1-\sin^2\theta_f = 1 - \frac{2H}{R} + \frac{H^2}{R^2}$$

$$\Rightarrow \sin\theta_f = \left(\frac{2H}{R} - \frac{H^2}{R^2}\right)^{1/2}$$

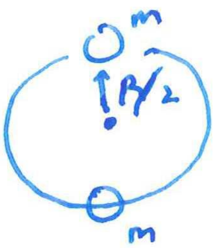
so:

$$W_{total} = F_0 R \frac{H}{R}\left[\frac{2R}{H} - 1\right]^{1/2}$$

$$= \boxed{F_0 H\sqrt{\frac{2R}{H} - 1}}$$

Problem 4

(a)



$$\frac{mv^2}{R/2} = \frac{Gm^2}{R^2}$$

$$\Rightarrow v^2 = \frac{Gm}{2R}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{Gm}{2R}}}$$

(b)

$$E_{Total} = KE + PE$$

$$= \frac{mv^2}{2} + \left(-\frac{Gm^2}{R}\right)$$

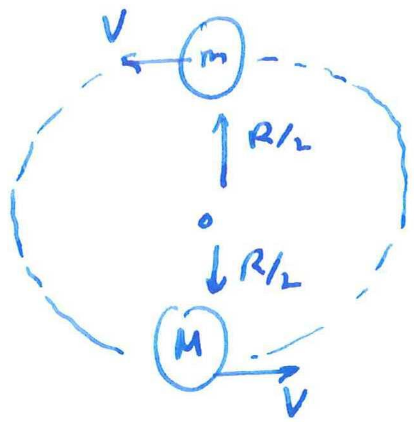
$$= \frac{m}{2}\frac{Gm}{2R} - \frac{Gm^2}{R}$$

$$= \frac{Gm^2}{4R} - \frac{Gm^2}{R}$$

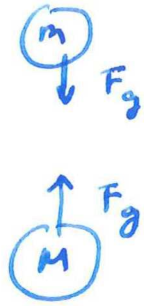
$$= \boxed{-\frac{3Gm^2}{4R}}$$

(c) $\boxed{\text{No}}$

Reason :



Say $m \neq M$.



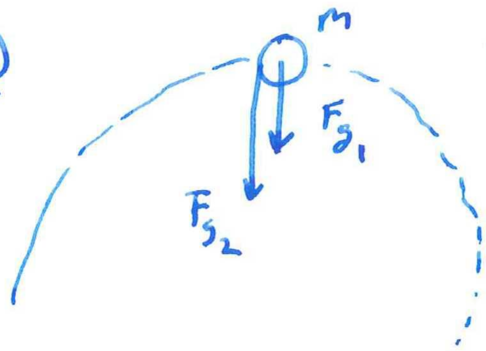
$$F_g = \frac{GMm}{R^2}$$

- Centripetal force on m : $\frac{mV^2}{R/2} = \frac{GMm}{R^2} \Rightarrow V^2 = \frac{GM}{2R}.$
- Centripetal force on M : $\frac{MV^2}{R/2} = \frac{GMm}{R^2} \Rightarrow V^2 = \frac{Gm}{2R}$

Not equal because $M \neq m$.

So this type of orbit cannot occur.

(d)



$$F_{g_1} = \frac{GMm}{R^2} \qquad F_{g_2} = \frac{Gm^2}{4R^2}$$

so $F_{cent} = \frac{mV^2}{R}$

and $\frac{\cancel{m}V^2}{\cancel{R}} = \frac{GM\cancel{m}}{R^{\cancel{2}}} + \frac{Gm^{\cancel{2}}}{4R^{\cancel{2}}}$

$$\Rightarrow V^2 = \frac{GM}{R} + \frac{Gm}{4R}$$

$$= \frac{4GM + Gm}{4R}$$

$$= \frac{G(4M+m)}{4R}. \Rightarrow V = \sqrt{\frac{G(4M+m)}{4R}}$$

$$VT = 2\pi R$$

$$\Rightarrow T = \frac{2\pi R}{V}$$

$$= 2\pi R\sqrt{\frac{4R}{G(4M+m)}}$$

$$= \boxed{4\pi\sqrt{\frac{R^3}{G(4M+m)}}}$$

Problem 5

(a)
$$V_{avg} = \frac{V\Delta T + 0\cdot\Delta T + V\Delta T + 0\cdot\Delta T + \dots + V\Delta T + 0\Delta T}{2N\Delta T}$$
$$= \frac{\cancel{N} V \cancel{\Delta T}}{2\cancel{N}\cancel{\Delta T}}$$
$$= \boxed{\frac{V}{2}}$$

(b) $R\omega_{avg} = V_{avg}$.
$$\Rightarrow \boxed{\omega_{avg} = \frac{V}{2R}}$$

(c) During time duration ΔT, while car moves at speed V, the wheel turns $\Delta\theta = R\omega$

$\uparrow$ Angle turned (in radians)

so after time $2N\Delta t$: $N\cdot\Delta\theta = NR\omega$ ← total angle turned

So: $$\frac{2\pi R}{N\Delta\theta} = \frac{2\pi \cancel{R}}{N\cancel{R}\omega} \Rightarrow \boxed{\frac{2\pi}{N\omega}}$$

↑ total # turns of wheels

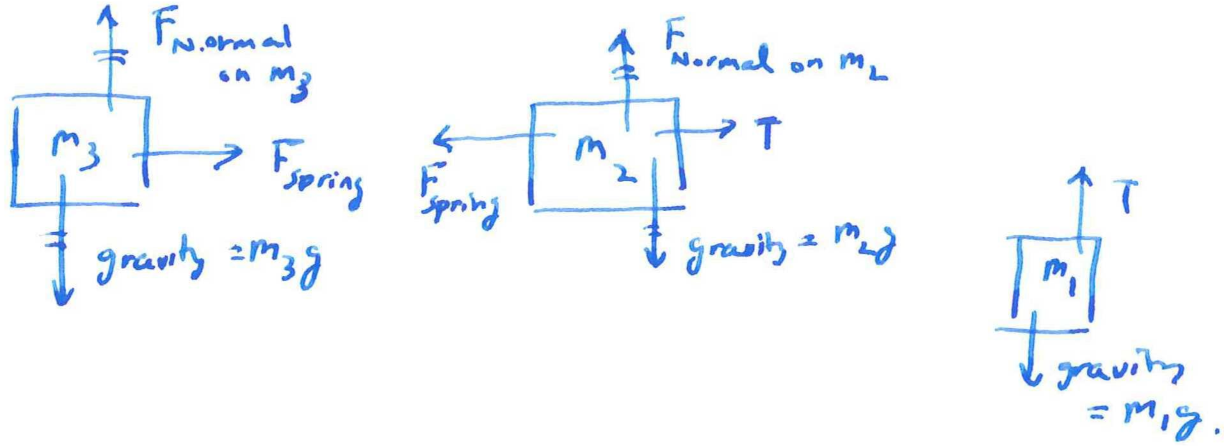
(d) Work done = KE gained
$$= \boxed{\frac{mv^2}{2}}$$

(e) Every time that light is turned off, car stops ⇒ KE → 0.

So, after N turning offs of light, $\boxed{\frac{Nmv^2}{2}}$ ← total energy lost to heat.

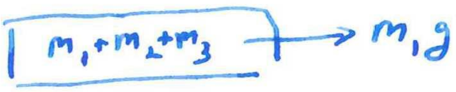
Problem 6

(a)



(b) The net external force is $m_1 g$.

It acts as if:



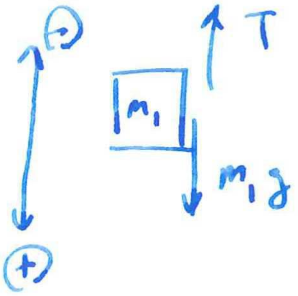
So: $(m_1+m_2+m_3)\, A_{\text{system}} = m_1 g$.

$$\Rightarrow \boxed{A_{\text{system}} = \frac{m_1 g}{m_1+m_2+m_3}}$$

← magnitude of acceleration

- m_1 accelerates down
- m_2, m_3 accelerate to right.

(c)

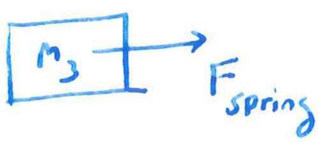


$$m_1 A_{\text{system}} = m_1 g - T$$

$$\Rightarrow T = m_1\left(g - \frac{m_1 g}{m_1+m_2+m_3}\right)$$

$$= \boxed{m_1\left(\frac{(m_2+m_3)\, g}{m_1+m_2+m_3}\right)}$$

(d)



$$F_{\text{spring}} = m_3 A_{\text{system}} \Rightarrow K\Delta x = \frac{m_3 m_1 g}{m_1+m_2+m_3}$$

$$\Rightarrow \boxed{\Delta x = \frac{m_1 m_3 g}{K(m_1+m_2+m_3)}}$$

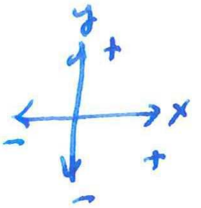
Spring is stretched

(e) Bonus

Let $M_{total} = m_1 + m_2 + m_3$

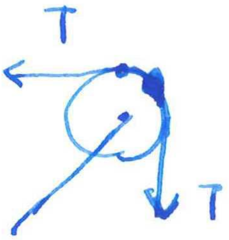
$$M_{total}\vec{A}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 \qquad \vec{a}_3 = \vec{a}_2$$

$$= \left((m_2+m_3)A_{system}, -m_1 A_{system}\right) \qquad \downarrow \vec{a}_1$$



$$\Rightarrow \vec{A}_{CM} = \boxed{\left(\frac{(m_2+m_3)m_1 g}{M_{total}^2}, \frac{-m_1^2 g}{M_{total}^2}\right)}$$

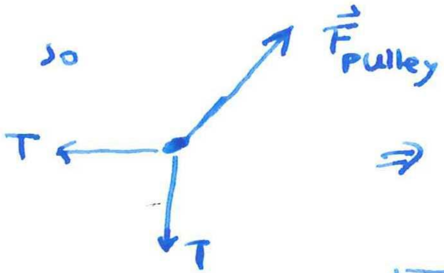
(f)



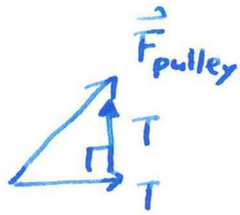
Massless pulley

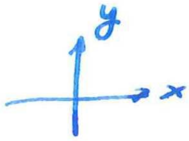
so $F_{Net} = 0$.

so



$\Rightarrow$





$$\Rightarrow \boxed{\vec{F}_{pulley} = (T, T)}$$

You can check that this is the right answer (and also derive this answer) by seeing that

$$\boxed{M_{total}\vec{A}_{CM} = \vec{F}_{pulley} + (0, -m_1 g)}$$