Physics 1A for NB

Retake exam (full)

May 4, 2016, 9:00-12:00h

The exam consists of five problems. Make each problem on a separate answer sheet, and hand the sheets in separately. Always show your work, and give full calculations / derivations / arguments.

1 Testing your knowledge (6 points)

NB: When applicable, always explain your answers!

(a) For each of the following laws, indicate under which condition(s) (if any) they're valid: Conservation of energy, conservation of linear momentum, and conservation of angular momentum.

(b) Sketch the position vs. time curve for a block attached to a spring that oscillates back-and-forth as a simple harmonic oscillator without friction. You can assume equilibrium position is at $x = 0$ cm and which is released at $t = 0$ s at $x = 10$ cm with speed zero.

(c) Give the magnitude and direction of the rotation vector of the Earth for the rotation that causes the day-night pattern (i.e. not the slower rotation of Earth around the sun).

(d) Sketch the phase diagram of methane. Indicate all phases and relevant points. Don't forget to label your axes.

Answers:

(a)
- Conservation of energy: all forces must be conservative, i.e., the work done by the force moving between two points must be independent of the path chosen [0.5 pt].
- Conservation of linear momentum: there must be no external forces acting on the system [0.5 pt].
- Conservation of angular momentum: there must be no external torques acting on the system [0.5 pt].

(b) See figure 1 [1 pt].

(c) The Earth rotates about its North-South axis. The Sun rises in the East, so as seen from above the North Pole, the Earth rotates counter-clockwise (to see this, consider a stationary Sun and a rotating Earth). The rotation vector of the Earth thus points upwards along the South to North Pole axis [0.5 pt], and has a magnitude of one revolution per day, or $1/(24 \cdot 60 \cdot 60) = 1/86400 = 1.16 \cdot 10^{-5}$ Hz [0.5 pt].

(d) See figure 2 [2.5 pt; subtract 0.5 pt each for missing axis labels, triple or critical point, phases, wrong lines between phases, and not having the solid/gas line end up in the origin].

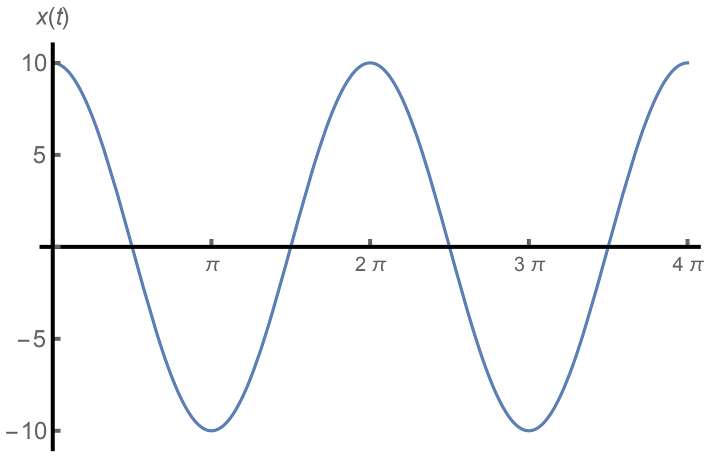

Figure 1: The position-time graph of an undamped harmonic oscillator for problem 1b.

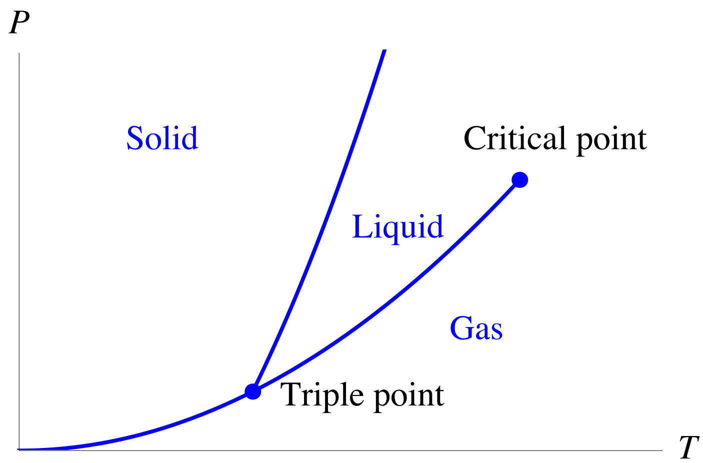

Figure 2: The phase diagram of methane (problem 1d).

2 Energy of an electron - 7 points

The potential energy of an electron in a hydrogen atom is given by

$$U(r) = -\frac{a}{r} + \frac{b}{r^2}. \tag{1}$$

Here a and b are positive constants and r is the distance to the origin (where the nucleus of the atom is located).

(a) Give the dimensions of a and b.

(b) Does this potential energy produce an attractive or a repulsive force at small distances? And at large distances? (Hint: Sketch the U(r) as a function of r)

(c) Find the equilibrium point(s) of this potential energy and determine whether they are stable equilibrium or unstable equilibrium points.

(d) An electron is released at $r = \infty$ with speed zero. Determine the maximum speed the electron can get. (Hint: Total energy of an electron is the kinetic energy plus the potential energy U(r). What is U(r) when r approaches infinity?)

(e) For the electron in (d) that is released from $r = \infty$, find the minimum distance from the nucleus (nucleus is at r=0) that the electron can travel to.

Answers:

(a) The dimension of U is energy, or force times distance, which is ML^2T^{-2}. De dimension of a is energy times distance, so $[a] = ML^3T^{-2}$ [0.5 pt]; de dimension of b is energy times distance squared, so $[b] = ML^4T^{-2}$ [0.5 pt].

(b) *Method 1*: The force is minus the derivative of the potential energy, so

$$F(r) = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{a}{r^2} + \frac{2b}{r^3}.$$

For small values of r the second term dominates, so $F(r)$ is positive and the force is repulsive; for large values of r the first term dominates, so $F(r)$ is negative and the force is attractive [1 pt].
Method 2: We sketch the potential energy, see figure 3. From the sketch, we see that for small values of r the slope of the potential energy is negative, so the force is repulsive, and for large values of r the slope of $U(r)$ is positive, so the force is attractive [1 pt].

(c) To identify the equilibrium points, we look for positions where the force vanishes, i.e. where the derivative of the potential is identically zero:

$$0 = \frac{\mathrm{d}U}{\mathrm{d}r} = \frac{a}{r^2} - \frac{2b}{r^3} \quad \Rightarrow \quad r = \frac{2b}{a}$$

[1 pt]. To determine the stability of this point, we consider the second derivative of the potential energy:

$$\left.\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right|_{r=2b/a} = -\frac{2a}{r^3} + \left.\frac{6b}{r^4}\right|_{r=2b/a} = \frac{a^4}{8b^3} > 0.$$

Because the second derivative is positive, the equilibrium point is stable. Because this is moreover the only equilibrium point, this is the global minimum and the equilibrium point is globally stable [1 pt]. *Alternative*: for the stability of the equilibrium point we can also look at the graph of the potential energy, figure 3, from which we immediately see that the equilibrium point corresponds to the global minumum of $U(r)$, and therefore is a stable equilibrium point [1 pt].

(d) At $r = \infty$ the potential energy equals $U(\infty) = 0$, so the total energy of the particle is $E = K + U = 0$. The total energy is conserved, so the kinetic energy is maximal when the potential energy is minimal, which happens at the global minimum of $U(r)$ at $r = 2b/a$: $U(2b/a) = -a^2/4b$ [1 pt]. The maximum kinetic energy is therefore $a^2/4b$, and the maximum speed follows from $K_{\mathrm{max}} = \frac{1}{2}mv_{\mathrm{max}}^2 = a^2/4b$, so $v_{\mathrm{max}} = a/\sqrt{2mb}$ [1 pt].

(e) The minimum distance to the nucleus (at the origin) is reached at the point where the kinetic energy is (again) zero, and the potential energy equals that of the starting point (here $r = \infty$ en $U(\infty) = 0$) [0.5 pt]. We therefore need to solve $U(r) = 0$, so:

$$0 = U(r) = -\frac{a}{r} + \frac{b}{r^2} \quad \Rightarrow \quad r = \frac{b}{a}$$

[0.5 pt].

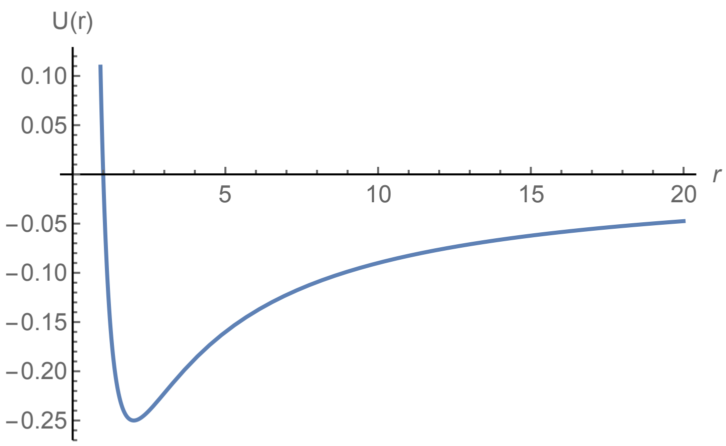

Figure 3: The potential energy of problem 2 (here with $a = b = 1$). NB: The shape of the graph is the same for any choice of a en b as long as they are both positive.

3 Mechanics - 8 points

(a) A shell is shot with an initial velocity of 25 m/s, at an angle of 50° with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, drops to the ground vertically. How far from the gun does the other fragment land (assuming no air drag and level terrain)?

(b) A uniform sphere of radius R is supported by a rope attached to a vertical wall, as shown in figure 4. The rope joins the sphere at a point where a continuation of the rope would intersect a horizontal line through the sphere's center a distance $R/2$ beyond the center, as shown in figure 1. What is the smallest possible value for the coefficient of friction between wall and sphere?

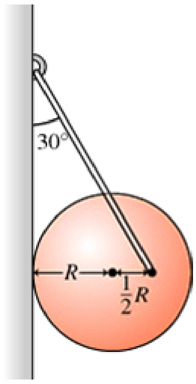

Figure 4: A ball resting on a wall (problem 3b).

(c) A proton (mass 1 u, i.e. mass of one proton) moving at $v_1 = 6.90 \cdot 10^6$ m/s collides elastically and head-on with a second particle moving in the opposite direction at $v_2 = 2.80 \cdot 10^6$ m/s. After the collision, the proton is moving opposite to its initial direction at $8.62 \cdot 10^6$ m/s. Find the mass (in unit of u) and final velocity of the second particle.

Answers:

(a) We split this problem into two parts. First we calculate the horizontal distance traveled while going upward. The second half is after the break up. From momentum conservation we calculate the new velocity and from that we can and the distance traveled while descending.

(1) Ascending: We can find the time t_1 it takes to reach the highest point h_{max} from

$$\begin{aligned} v_y(t) &= v_y(0) - gt \\ 0 &= v \sin\theta - gt \\ t_1 &= \frac{v\sin\theta}{g} \end{aligned}$$

[0.5 pt]. The distance traveled is then

$$x_1 = v_x(0)t = v\cos\theta t_1 = \frac{v^2}{g}\cos\theta\sin\theta$$

[0.5 pt].

(2) Descending: The shell (mass $2m$) breaks apart into two pieces of equal mass m. Before the breakup, the shell (at the top) has a velocity $\vec{v} = v_x(0)\hat{x} = v\cos\theta\hat{x}$. Conservation of momentum in the x-direction then gives:

$$\begin{aligned} 2mv\cos\theta &= mv_{x,1} + mv_{x,2} \\ 2v\cos\theta &= v_{x,2} \end{aligned}$$

as $v_{x,1} = 0$ [0.5 pt]. In the absence of air resistance, the time t_1 it takes to reach the highest altitude is the same as the time t_2 it takes to fall [0.5 pt]. The distance traveled in the fall equals $x_2 = v_{x,2}t_2$. The total distance traveled is thus:

$$x = x_1 + x_2 = v\cos\theta t_1 + 2v\cos\theta t_2 = 3\frac{v^2}{g}\cos\theta\sin\theta = 94 \text{ m}$$

[0.5 pt].

(b) There are four forces: F_z, F_f, F_N and T (in the rope). We write down force balance in the x and y direction, and torque balance about the point where the line of T intersects the horizontal (could take a different point here, same answer of course). These give us:

$$\begin{aligned} F_N &= T\sin\theta, \\ F_z &= F_f + T\cos\theta, \\ F_f\frac{3}{2}R &= F_z\frac{1}{2}R. \end{aligned}$$

[0.5 pt each]. The third equation gives $F_f = \frac{1}{3}F_z = \frac{1}{3}mg$. Substituting in the second equation, we get $T = (F_z - F_f)/\cos\theta = \frac{2}{3}mg/\cos\theta$ [0.5 pt]. Substituting that in the first equation, we find $F_N = \frac{2}{3}mg\tan\theta$ [0.5 pt]. Since $F_f \leq \mu F_N$, the smallest possible value of μ is $\mu = F_f/F_N = \frac{1}{2}\cot(30°) = \frac{1}{2}\sqrt{3}$ [0.5 pt].

(c) The collision is elastic, so we have conservation of both momentum and kinetic energy (labeling the velocities of the particles after the collision v_3 and v_4):

$$\begin{aligned} m_1v_1 + m_2v_2 &= m_1v_3 + m_2v_4, \\ m_1v_2^2 + m_2v_2^2 &= m_1v_3^2 + m_2v_4^2. \end{aligned}$$

[0.5 pt]. Rewriting these to get all the m_1 terms on the left, and m_2 terms on the right, we get:

$$\begin{aligned} m_1(v_1 - v_3) &= m_2(v_4 - v_2), \\ m_1(v_1^2 - v_3^2) &= m_2(v_4^2 - v_2^2). \end{aligned}$$

[0.5 pt]. We can factor the second line to $m_1(v_1 - v_3)(v_1 + v_3) = m_2(v_4 - v_2)(v_4 + v_2)$, so we can divide the kinetic-energy equation by the momentum equation to obtain $v_1 + v_3 = v_2 + v_4$ [0.5 pt]. Since we know three of these, getting the fourth (v_4) is easy; we can then substitute everything back in the momentum equation to get m_2 (note the sign convention for the velocities: leftward is negative):

$$\begin{aligned} v_4 &= v_1 + v_3 - v_2 = (6.9 - 8.62 + 2.8) \cdot 10^6 = 1.08 \cdot 10^6 \text{ m/s}, \\ m_2 &= \frac{v_1 - v_3}{v_4 - v_2}m_1 = \frac{6.9 + 8.62}{1.08 + 2.8}(1 \text{ u}) = 4 \text{ u}. \end{aligned}$$

[0.5 pt each].

4 Rotating objects - 8 points

Two children with mass $m_1 = 10$ kg and $m_2 = 10$ kg sit on a simple merry-go-round that can be described as a solid disk of 100 kg with a radius of 2.0 m. The merry-go-round is free to rotate about its center, and initially does so with a frequency ω_0 of 5.0 revolutions per minute. A third child with mass $m_3 = 10$ kg runs towards the merry-go-round with a speed v_0 of 1.0 m/s, and under an angle of 30° with the tangent to the merry-go-round (see figure 5a). When arriving at the merry-go-round, the third child jumps on it, and afterwards spins around with the other two.

(a) Find the rotational velocity of the merry-go-round after the third child jumped on it.

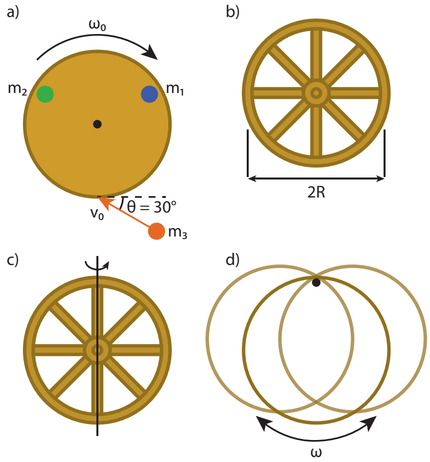

Figure 5: Four rotating systems. a) Three children on a merry-go-round. b) and c) Wagon wheel. d) Hula hoop on a peg.

A wagon wheel is constructed as shown in figure 5b. The radius of the wheel is R. Each of the spokes that lie along the diameter has a mass m, and the rim has mass M (you may assume the thickness of the rim and spokes are negligible compared to the radius R).

(b) What is the moment of inertia of the wheel about an axis through the center, perpendicular to the plane of the wheel?

(c) For the same wheel as in (b), what is the moment of inertia for an axis through the center and two of the spokes, in the plane of the wheel (figure 5c)?

(d) A hula hoop of mass M and radius R hangs from a peg. Find the period of the hoop as it gently rocks back and forth on the peg (figure 5d).

Answers:

(a) The key to this problem is conservation of angular momentum [0.5 pt; also give these points if angular momentum conservation is not explicitly mentioned but is used correctly]. The total angular momentum before and after the third child's jump are equal, and given by:

$$
\begin{aligned}
L_{\text{voor}} &= L_{\text{schijf}} + L_1 + L_2 + L_3 \\
&= I_{\text{schijf}}\omega_0 + m_1 R^2 \omega_0 + m_2 R^2 \omega_0 + m_3 v_0 \cos(\theta) R \\
&= \left(\frac{1}{2}M + m_1 + m_2\right) R^2 \omega_0 + m_3 v_0 R \cos(\theta) \\
L_{\text{na}} &= L_{\text{schijf}} + L_1 + L_2 + L_3 \\
&= \left(\frac{1}{2}M + m_1 + m_2 + m_3\right) R^2 \omega_1 \\
\omega_1 &= \frac{\left(\frac{1}{2}M + m_1 + m_2\right)\omega_0 + m_3 (v_0/R)\cos(\theta)}{\frac{1}{2}M + m_1 + m_2 + m_3} \\
&= \frac{(50+10+20)\cdot(5.0/60) + 10\cdot(1.0/2.0)\cdot\frac{1}{2}\sqrt{3}}{50+10+20+10} = 0.12\ \text{s}^{-1} = 7.3\ \text{rpm}.
\end{aligned}
$$

[1.0 pt for the correct expression for L_{voor}, 0.5 pt for he correct expression for L_{na}, 1.0 pt for the correct calculation of the new rotational velocity (in rotations per second (s^{-1}) of per minute (rpm))].

(b) The wheel consists of two parts: a ring of mass M and radius R, which has a moment of inertia $I_{\text{ring}} = MR^2$ about its center [0.5 pt], and eight spokes of mass m and length R, which are rotated about their end, with moments of inertia $I_{\text{spoke}} = \frac{1}{3}mR^2$ [0.5 pt]. The total moment of inertia is $I_{\text{wheel}} = I_{\text{ring}} + 8I_{\text{spoke}} = (M + 8m/3)R^2$ [0.5 pt].

(c) We invoke the perpendicular axes theorem: the moment of the wheel about an axis through the center, perpendicular to its plane ($I_z = I_{\text{wheel}}$), equals the sum of the moments of inertia about two perpendicular axes in the plane, e.g. the one shown (which we'll call I_x) and the one perpendicular to it (I_y) [0.5 pt]. Since by symmetry I_x and I_y are identical in this case [0.5 pt], we have $I_z = I_x + I_y = 2I_x$ and $I_x = I_z/2 = (\frac{1}{2}M + \frac{4}{3}m)R^2$ [0.5 pt - Naturally, a direct calculation is also allowed, if the found answer is correct].

(d) The frequency of a pendulum is given by $\omega_0 = \sqrt{MgL/I}$ [0.5 pt]. We need the moment of inertia of the hoop around a point on its edge. For this, we use the parallel-axis theorem: $I = I_{\text{CM}} + Md^2$, with d the distance to the center of mass [0.5 pt]. The hoop has a moment of inertia of MR^2 around its center, and thus a moment of inertia of $MR^2 + MR^2 = 2MR^2$ around a point on its edge [0.5 pt]. For the period we get:

$$
T = 2\pi/\omega_0 = 2\pi\sqrt{\frac{2MR2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}
$$

[0.5 pt].

5 Thermodynamics - 11 points

We consider an ideal gas with adiabatic exponent $\gamma = 4/3$.

(a) Find the volume specific heat C_V of this gas. Hint: C_V is not $(3/2)R$, since $\gamma \neq 5/3$. Here, you can use the fact that for any ideal gas, the difference between the volume specific heat C_V and the pressure specific heat C_P is $C_P - C_V = R$.

(b) How many atoms are in one molecule of this gas?

We take a sample of this gas that occupies a volume of 5.00 liters, at a temperature of 300 K and a pressure of 100 kPa. The gas is compressed adiabatically to 1/5 of its original volume. Next, its temperature is brought back to 300 K while holding the volume constant. Finally, the gas isothermally expands back to its original volume.

(c) Sketch the pV diagram of the cycle. Indicate any important points, and make sure to put them at the right positions (i.e. calculate symbolically any values of p and V, and write down your calculations). Don't forget to properly label your axes. Indicate the direction of the cycle with arrows.

(d) In the second step of the cycle, do you need to cool down or heat up the gas to let it return to its original temperature of 300 K? Explain your answer.

(e) Find the work done on the gas in the entire cycle.

(f) Could this cycle be used as a heat engine? If so, calculate its efficiency. If not, find another application that it could be used for, and calculate its associated coefficient of performance (COP = (what we get out)/(what we put in)).

(g) What should the total change in entropy of the cycle be? Explain your answer.

(h) By directly calculating, find the change in entropy in each of the three steps. (Note: obviously, you can check your answer by summing the contributions of the individual steps to get the total change in (g). However, here you should calculate them explicitly, not use what you know about the total change).

Answers:

(a) For any ideal gas, $C_p = C_V + R$, and $\gamma = C_p/C_V = (C_V + R)/C_V$. Substituting $\gamma = 4/3$ gives $C_V = 3R$ [1 pt].

(b) By the equipartition theorem, we have $\frac{1}{2}k_B T$ per degree of freedom. By definition of C_V, we have $C_V = (1/n)\,\mathrm{d}U/\,\mathrm{d}T = ((\#\mathrm{dof})/2)(k_B/n) = \frac{1}{2}(\#\mathrm{dof})R$ [0.5 pt]. Therefore, $C_V = 3R$ implies 6 degrees of freedom, which corresponds to a triatomic molecule [0.5 pt] (three translational and three rotational dofs).

(c) $V_1 = 5.00$ L, $V_2 = V_3 = V_1/5 = 1.00$ L. Process $1 \to 2$ is adiabatic, so $p_2 = (V_1/V_2)^\gamma p_1 = 855$ kPa [0.5 pt]. Process $3 \to 1$ is isothermal, so $p_3 = (V_1/V_3)p_1 = 5p_1 = 500$ kPa [0.5 pt]. The diagram is shown in figure 6. [1 point total for the diagram: 0.5 for correct placement of points 1, 2 and 3, and 0.5 for correct drawing of the lines and arrows. Subtract 0.5 if axis labels or units are missing.]

(d) *Method 1*: direct calculation. $T_3 = T_1$ (given), find T_2 through ideal gas law: $p_3/T_3 = p_2/T_2$, so $T_2 = (p_2/p_3)T_3 = 513$ K. Therefore, you have to cool down going from 2 to 3 [1 pt].
Method 2: Simply note that same volume, higher pressure implies higher temperature, so $T_2 > T_3$, and you need to cool down going from 2 to 3 [1 pt].
Method 3: In an adiabatic compression, there is no heat exchange with the environment. Hence all work done on the gas must result in an increase in internal energy, which scales linearly with the temperature. Hence the temperature increases, so $T_2 > T_3$, and you need to cool down going from 2 to 3 [1 pt].
Method 4: In an isothermal compression, temperature is constant, and pressure increases according to $pV =$ constant. In an adiabatic compression, there is no heat exchange with the environment, and pressure increases according to $pV^\gamma =$ constant. As $\gamma > 1$, this curve is steeper than the isothermal one (see also sketch at c). Consequently, point 2 lies on a different isotherm than point 1 and 3, with a higher associated temperature, so $T_2 > T_3$, and you need to cool down going from 2 to 3 [1 pt].

(e) We use that $W = -\int p \, \mathrm{d}V$.
Process $1 \to 2$ is adiabatic, so $pV^\gamma = c$, and $p = p_1(V_1/V)^\gamma$, so

$$W_{12} = p_1 V_1^\gamma \frac{1}{\gamma - 1}\left(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}}\right) = 1065 \text{ kJ}$$

(or $W_{12} = (p_2V_2 - p_1V_1)/(\gamma - 1) = 1065$ kJ) [1 pt].
Process $2 \to 3$ is isovolumetric, so $\mathrm{d}V = 0$ and hence $W_{23} = 0$ [0.5 pt].
Process $3 \to 1$ is isothermal, and the work done on the gas is negative, because the gas does work when expanding. To calculate the work, we use the ideal gas law, $pV = nRT$, with the rhs being constant here. Consequently $p = p_1V_1/V$, and $W_{31} = -p_1V_1 log(V_1/V_3) = -805$ kJ [0.5 pt].
The work done on the gas in the entire cycle is thus 260 kJ [0.5 pt - subtract only 0.5 point if sign in W_{31} incorrect but summed correctly to get final answer].

(f) Since it requires work to carry out this cycle, it cannot be used as an engine [0.5 pt].
Option 1: The cycle could be used as an engine when run in reverse. In that case, the efficiency is given by W/Q_h, where Q_h is the heat absorbed. Heat is rejected to a cool reservoir during isothermal compression (process $1 \to 3$), and absorbed from a hot reservoir during isovolumetric heating (process $3 \to 2$). In the latter process,

$$Q = \Delta E = nC_V\Delta T = n3R(T_2 - T_3) = 3p_1V_1\frac{T_2 - T_3}{T_1} = 1065 \text{ kJ}.$$

Therefore, the efficiency is 0.244 [1 pt].
Option 2: The cycle could be used as a refrigerator, extracting heat from some sample in process $3 \to 1$ [0.5 pt]. What we want is Q_c, the amount of heat extracted, which for an isothermal process is $-W$, so $Q_c = 805$ kJ. What we put in is work done on the gas, $W = 260$ kJ, so the COP equals $805/260 = 3.10$ [0.5 pt].
Option 3: The cycle could be used as a heat pump, delivering heat to some sample in process $2 \to 3$ [0.5 pt]. The amount of heat delivered is the same Q_h as calculated

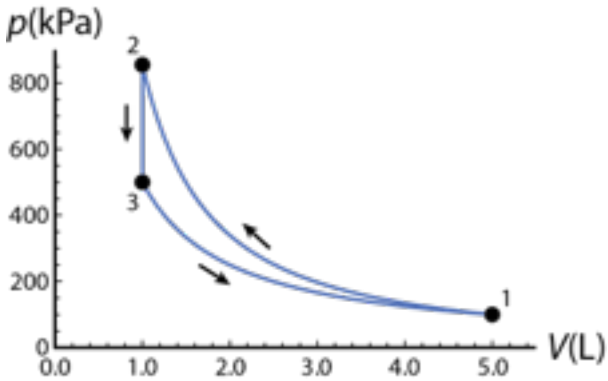

Figure 6: pV diagram of problem 5c.

in option 1, $Q_h = 1065$ kJ, and the amount of work done is still the total amount of work done in the cycle, $W = 260$ kJ, so the COP equals $1065/260 = 4.10$ [0.5 pt].

(g) Since the processes are all reversible, and the cycle is closed, the total change of entropy should be zero [0.5 pt].

(h) We use $\Delta S = \int \mathrm{d}Q/T$.
In the adiabatic process $1 \to 2$, $\mathrm{d}Q = 0$, and thus $\Delta S_{12} = 0$ [0.5 pt].
In the isovolumetric process $2 \to 3$, $\mathrm{d}Q = nC_V\,\mathrm{d}T$, $n = p_1V_1/RT_1$, $C_V = 3R$, so $\mathrm{d}Q = (3p_1V_1/T_1)\,\mathrm{d}T$. The entropy change thus equals

$$\Delta S = 3\frac{p_1V_1}{T_1}\int \frac{1}{T}\,\mathrm{d}T = \frac{p_1V_1}{T_1}\log\left(\frac{T_3}{T_2}\right) = -2.68 \text{ kJ}$$

[0.5 pt].
In the isothermal process $3 \to 1$, T is constant, so $\Delta E = 0$, $\Delta Q = -W$, and $\Delta S = (1/T)\int \mathrm{d}Q = \Delta Q/T = -W/T = 2.68$ kJ [0.5 pt].
[NB: Naturally, these add up to zero. As explicitly stated in the question, that fact may not be used to calculate one of the values of ΔS (so if they do, no points for that part).]